Chapter 2 - Integration techniques

2.1 Trigonometric integrals

2.1.1 Integrating powers of sine and cosine

Consider integrals of the form:

$$\int \cos^m ax \sin^n bx \, dx\,, \qquad m, n \in \mathbb{Z}^+$$

To evaluate these integrals, we use substitutions based on the following cases:

Case	*Substitution*
m is odd	$u = \sin ax$
n is odd	$u = \cos bx$
m, n both odd	$u = \cos ax \;\; or \sin bx$
m, n both even	Use a combination of the above and ii

Important identities:

i. $\sin^2 x + \cos^2 x = 1$

ii. $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$

2.1.2 Integrating powers of sine and cosine

Consider integrals in the form:

$$\int \cos mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx, \quad \int \sin mx \sin nx \, dx, \quad m, n \in \mathbb{R}$$

To evaluate these integrals, use the product to sum identities:

i. $\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

ii. $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

iii. $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos\,(A+B))$

2.1.2 Integrating powers of tan and cosec

To evaluate these integrals, use the following identities:

i. $\tan^2 x + 1 = \sec^2 x$

ii. $\frac{d}{dx}\tan x = \sec^2 x$

iii. $\frac{d}{dx}\sec x = \sec x \tan x$

2.2 Reduction formulae

Reduction is best demonstrated via examples:

Example

Suppose I_n is defined by

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \ dx\,, \qquad n \geq 0$$

Show that:

$$I_n = \frac{1}{n-1} - I_{n-2}, \qquad \forall n \geq 2$$

Hence evaluate

$$\int_0^{\frac{\pi}{2}} \tan^6 x \, dx$$

Solution

Using $\tan^2 x + 1 = \sec^2 x$ and splitting the integral

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2} x \tan^2 x \ dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \,(\sec^2 x - 1) \ dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \ dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

Using, $u = \tan x$ and $du = \sec^2 x$

$$= \int_0^{\frac{\pi}{4}} u^{n-2} \ du - I_{n-2}$$

$$= \left[\frac{u^{n-1}}{n-1}\right]_0^1 - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

Therefore we conclude:

$$I_n = \frac{1}{n-1} - I_{n-2}$$

Now to solve,

$$\int_0^{\frac{\pi}{2}} \tan^6 x \, dx$$

which is just I_6,

$$I_6 = \frac{1}{6-1} - I_{6-2}$$

$$= \frac{1}{5} - I_4$$

$$= \frac{1}{5} - \left(\frac{1}{3} - I_2\right)$$

$$= \frac{1}{5} - \frac{1}{3} + \left(\frac{1}{1} - I_o\right)$$

$$= \frac{1}{5} - \frac{1}{3} + \frac{1}{1} - \int_0^{\frac{\pi}{4}} dx$$

$$= \frac{1}{5} - \frac{1}{3} + \frac{1}{1} - \frac{\pi}{4}$$

$$= \frac{13}{15} - \frac{\pi}{4}$$

Example

Suppose

$$I_n = \int \sin^n x \ dx\,, \qquad n \geq 0$$

Show that:

$$I_n = -\frac{\sin^{n-1}\cos x}{n} + \frac{n-1}{n} I_{n-2} \quad \forall n \geq 2$$

Solution

Splitting the integral:

$$I_n = \int \sin^{n-1} x \sin x \ dx$$

Applying integration by parts:

$u = \sin^{n-1} x$	$dv = \sin x \, dx$
$du = (n-1)\sin^{n-2} x \cos x \; dx$	$v = -\cos x$

$$I_n = \int \sin^{n-1} x \sin x \; dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \; dx$$

$$= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$$

Gathering I_n terms to the LHS,

$$nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$$

$$I_n = \frac{1}{n}(-\sin^{n-1} x \cos x + (n-1)I_{n-2})$$

as required.

2.3 Trigonometric and hyperbolic substitutions

The following table indicates which substitution is used for integrals containing expressions $\sqrt{\pm x^2 \pm a^2}$

Expression	Trigonometric substitution	Hyperbolic substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$x = a\tanh\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$x = a\cosh\theta$

2.4 Integrating rational functions

A rational function f is of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials.

- f is proper if the degree of p is less than the degree of q
- f is improper is the degree of p is greater than or equal to the degree of q

- a quadratic polynomial is irreducible if it has no real linear factors. (a quadratic $ax^2 + bx + c$ is irreducible if its discriminant $b^2 - 4ac < 0$).

2.4.1. Common tactics

- A common tactic is to rewrite the integrand so that the derivative of the denominator is sitting on the numerator:

$$\begin{aligned}
\int \frac{x}{x^2+2x+10}dx &= \frac{1}{2}\int \frac{2x}{x^2+2x+10}\,dx \\
&= \frac{1}{2}\int \frac{2x+\mathbf{2}-\mathbf{2}}{x^2+2x+10}\,dx \\
&= \frac{1}{2}\int \frac{2x+2}{x^2+2x+10}\,dx - \frac{1}{2}\int \frac{2}{x^2+2x+10}\,dx \\
&= \frac{1}{2}\ln|x^2+2x+10| - \frac{1}{2}\int \frac{2}{x^2+2x+1+9}\,dx \\
&= \frac{1}{2}\ln|x^2+2x+10| - \int \frac{1}{(x+1)^2+9}\,dx \\
&= \frac{1}{2}\ln|x^2+2x+10| - \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C
\end{aligned}$$

2.4.2. Partial fraction decompositions

- It can be shown that every proper rational function Sometimes rational functions are in the form f can be written as a unique sum of functions of the form

$$\frac{A}{(x-a)^k} \quad and \quad \frac{Bx+C}{(x^2+bx+c)^k}$$

where $x^2 + bx + c$ is irreducible. This sum is called the partial fractions decomposition of f. To find the partial fractions decomposition of a proper rational function $\frac{p}{q}$, we factorise the denominator q as much as possible i.e. express q as product of real linear fractions and irreducible quadratic factors. The form of the partial fractions decomposition is determined by factorisation. There are several cases depending on the type of factorisation.

Case 1 - denominator splits into distinct linear factors

$$\frac{7x-1}{x^2-2x-3} = \frac{7x-1}{(x+1)(x-3)} = \frac{A}{x-3} + \frac{B}{x+1}$$

A, B can be found using the following method.

Multiply through by $(x-3)(x+1)$

$$7x-1 = A(x+1) + B(x-3) \quad \forall x \in \mathbb{R}$$

Pick suitable values of x to obtain A and B (commonly requires you to solve simultaneously etc)

$$x=-1 \Rightarrow \quad -8=-4B$$
$$x=3 \Rightarrow \quad 20=4A$$
$$A=5, B=2$$

so,

$$\frac{7x-1}{x^2-2x-3} = \frac{7x-1}{(x+1)(x-3)} = \frac{5}{x-3} + \frac{2}{x+1}$$

Case 2- denominator has a repeated linear factor

Consider:

$$\frac{x^2-3x+8}{x(x-2)^2}$$

Because it has a repeated linear factor $x-2$ in the denominator, we break it up as follows:

$$\frac{x^2-3x+8}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Then we multiply through by $x(x-2)^2$ and repeat the above process to obtain A, B, C

Case 3 - denominator has an irreducible quadratic factor

Break up such rational functions as follows and repeat the process. Examples below:

$$\frac{x^2+x}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$\frac{x^3-2x+4}{(x^2+5)(x^2+x+1)} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{x^2+x+1}$$

Case 4 - denominator has an irreducible quadratic factor

Break up such rational functions as follows and repeat the process. Examples below:

$$\frac{x^2+x}{(x^2+9)^3}=\frac{Ax+B}{x^2+9}+\frac{Cx+D}{(x^2+9)^2}+\frac{Ex+F}{(x^2+9)^3}$$

$$\frac{x^3-2x+4}{(x-2)(x^2+x+1)^2}=\frac{A}{x-2}+\frac{Bx+C}{x^2+x+1}+\frac{Dx+E}{(x^2+x+1)^2}$$

$$\frac{4x^4-3x^2+x-9}{x^3(x-7)(x^2+3)^2(x^2+x+2)}$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-7}+\frac{Ex+F}{x^3+3}+\frac{Gx+H}{(x^2+3)^2}+\frac{Ix+J}{x^2+x+2}$$

This is useful in integration where rational functions can be partially decomposed and then integrated with ease.

2.5 Other substitutions

Examples:

Pick a suitable substitution for each integral

a)

$$\int \frac{dx}{1+x^{\frac{1}{4}}}$$

- The aim is to replace the fractional power with something more convenient. The obvious substitution is $x=u^4$ leading to the substitution $dx=4u^3du$

b)

$$\int \frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}+x^{\frac{1}{4}}}dx$$

- The aim is to remove the fractional powers. The lowest common multiple of 2,3 and 4 is 12, so we choose the substitution $x=u^{12}$

c)

$$\int \frac{dx}{\sqrt{e^{2x}-1}}$$

- $u^2 = e^{2x} - 1$ is the most efficient substitution to remove the square root from the equation.
- $u^2 = e^{2x} - 1$ implies that

  $$2u\left(\frac{du}{dx}\right) = 2e^{2x} = 2(u^2+1)$$

  so

  $$dx = \frac{udu}{u^2+1}$$