# 7SD Solutions Series

***Worked Solutions to Popular Mathematics Texts***

*Suggested Worked Solutions to*

# *"4 Unit Mathematics"*

***( Text book for the NSW HSC by D. Arnold and G. Arnold )***

## *Chapter 8*
## *Harder 3 Unit Topics*



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# *7SD Solutions Series*

***Worked Solutions to Popular Mathematics Texts***

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Solutions are to "4 Unit Mathematics"

[ by D. Arnold and G. Arnold (1993), ISBN 0 340 54335 3 ]

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# Exercise 8.1

## 1 Solution

**(a)** IF $0<a<b$, then $2a<a+b$, $a<\dfrac{a+b}{2}$, and $a+b<2b$, $\dfrac{a+b}{2}<b$. Hence

$$a<\frac{a+b}{2}<b \text{ with equality iff } a=b.$$

**(b)** IF $0<a<b$, then $a^2<ab$, $a<\sqrt{ab}$, and $ab<b^2$, $\sqrt{ab}<b$. Hence

$$a<\sqrt{ab}<b \text{ with equality iff } a=b.$$

## 2 Solution

**(a)** It is clear that $(a+b)^2=(a-b)^2+4ab \Rightarrow (a+b)^2>4ab$ (equality iff $a=b$), since $(a-b)^2\geq 0$. Then, if $a>0$, $b>0$, we get

$$\frac{a+b}{2}>\sqrt{ab} \text{ (equality iff } a=b).$$

**(b)** If $a>0$, $b>0$, using $2ab<a^2+b^2$, we get

$$(a+b)^2=a^2+2ab+b^2<2a^2+2b^2, \text{ and } \left(\frac{a+b}{2}\right)^2<\frac{a^2+b^2}{2}.$$

Hence
$$\frac{a+b}{2}<\sqrt{\frac{a^2+b^2}{2}} \text{ (equality iff } a=b).$$

## 3 Solution

Consider

$$(ac-bd)^2-(a^2-b^2)(c^2-d^2)=a^2d^2-2acbd+b^2d^2=(ad-bc)^2\geq 0.$$

Hence $(a^2-b^2)(c^2-d^2)\leq(ac-bd)^2$ with equality iff $ad=bc$.

Consider

$(a^3-b^3)^2-(a^2-b^2)(a^4-b^4)=a^6-2a^3b^3+b^6-a^6+a^2b^4+a^4b^2-b^6=(ab^2-a^2b)^2\geq 0.$H

ence $(a^2-b^2)(a^4-b^4)\leq(a^3-b^3)^2$ with equality iff $a=b$.

### 4 *Solution*

If $a>0$, it is easily seen that

$$\left(a-\frac{1}{a}\right)^2=\left(a+\frac{1}{a}\right)^2-4a\frac{1}{a}\geq 0\Rightarrow\left(a+\frac{1}{a}\right)^2\geq 4\Rightarrow a+\frac{1}{a}\geq 2$$

with equality iff $a=1$. IF $a>0$, $b>0$ and $c>0$, using this inequality, we have

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})=1+1+1+(\frac{a}{b}+\frac{b}{a})+(\frac{b}{c}+\frac{c}{b})+(\frac{c}{a}+\frac{a}{c})\geq 3+2+2+2=9.$$

Hence

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\geq 9$$

with equality iff $a=b=c$. If we multiply the last inequality by $abc$, we deduce that

$$(a+b+c)(ab+bc+ca)\geq 9abc.$$

Since

$$(a+b+c)(ab+bc+ca)=a^2b+ab^2+a^2c+b^2c+bc^2+ac^2+3abc,$$

we get

$$a^2b+b^2c+c^2a+ab^2+bc^2+ca^2\geq 6abc$$

with equality iff $a=b=c$.

### 5 *Solution*

**(a)** It is easily seen that

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=a^3+ab^2+ac^2-a^2b-abc-ca^2$$
$$ba^2+b^3+bc^2-ab^2-b^2c-abc+ca^2+b^2c+c^3-abc-bc^2-c^2a=$$
$$=a^3+b^3+c^3-3abc.$$

**(b)** It is clear that

$$a^2+b^2\geq 2ab,$$
$$b^2+c^2\geq 2bc,$$
$$a^2+c^2\geq 2ac.$$

By addition

$$2(a^2+b^2+c^2)\geq 2(ab+bc+ca).$$

Hence $a^2+b^2+c^2\geq ab+bc+ca$ with equality iff $a=b=c$. Since $a+b+c$ is positive, and

$$a^2+b^2+c^2-ab-bc-ca\geq 0,$$

the right-hand side of identity in (**a**) is not negative. Hence

$$a^3+b^3+c^3\geq 3abc \text{ (equality iff } a=b=c\text{).}$$

Since $(x+y)^3=x^3+3x^2y+3xy^2+y^3$, we have

$$(a+b+c)^3=a^3+b^3+c^3+3a^2b+3ab^2+3a^2c+3b^2c+3ac^2+3bc^2+6abc.$$

Using the inequality from Example 4

$$a^2b+b^2c+c^2a+ab^2+bc^2+ca^2\geq 6abc,$$

and

$$a^3+b^3+c^3\geq 3abc,$$

we obtain

$$(a+b+c)^3\geq 27abc.$$

Hence

$$\frac{(a+b+c)}{3}\geq\sqrt[3]{abc} \text{ (equality iff } a=b=c\text{).}$$

After the substitution $a\to\frac{a}{b}$, $b\to\frac{b}{c}$, $c\to\frac{c}{a}$, the last inequality becomes

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3 \text{ (equality iff } a=b=c\text{).}$$

**6 *Solution***

(**a**) It is easily seen that

$$(a+b)^2\geq 4ab,$$
$$(b+c)^2\geq 4bc,$$
$$(a+c)^2\geq 4ac,$$

since $(a+b)^2=(a-b)^2+4ab\Rightarrow(a+b)^2>4ab$ (equality iff $a=b$). By multiplication, we get

$$(a+b)^2(b+c)^2(c+a)^2\geq 4^3a^2b^2c^2.$$

Hence $(a+b)(b+c)(c+a)\geq 8abc$ with equality iff $a=b=c$.

(**b**) IF $a>0$, $b>0$, $c>0$ and $d>0$, by using the inequality

$$\frac{(a+b+c)}{3}\geq\sqrt[3]{abc} \text{ (equality iff } a=b=c\text{)}$$

(see Example 5) with respect to the sums in the expression

$$(b+c+d)(a+c+d)(a+b+d)(a+b+c),$$

we have

$$a+b+c \geq 3\sqrt[3]{abc},$$
$$a+b+d \geq 3\sqrt[3]{abd},$$
$$a+c+d \geq 3\sqrt[3]{acd},$$
$$b+c+d \geq 3\sqrt[3]{bcd}.$$

By multiplication, we get

$$(b+c+d)(a+c+d)(a+b+d)(a+b+c) \geq 81abcd$$

with equality iff $a=b=c=d$.

### 7 *Solution*

**(a)** IF $a>0$, $b>0$ and $a+b=t$, it is easily seen that

$$(a+b)^2 \geq 4ab \Rightarrow \frac{a+b}{ab} \geq \frac{4}{a+b} \Rightarrow \frac{1}{a}+\frac{1}{b} \geq \frac{4}{t}.$$

Hence $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{t}$ with equality iff $a=b$.

**(b)** Consider the following expression:

$$(a+b)^2(\frac{1}{a^2}+\frac{1}{b^2}) = (a^2+2ab+b^2)(\frac{1}{a^2}+\frac{1}{b^2}) = 2+2(\frac{a}{b}+\frac{b}{a})+\frac{a^2}{b^2}+\frac{b^2}{a^2}.$$

By using the inequality $a+\frac{1}{a} \geq 2$ (see Example 4) with respect to the right-hand side, we get

$$(a+b)^2(\frac{1}{a^2}+\frac{1}{b^2}) \geq 2+4+2=8.$$

If $a+b=t$, hence

$$\frac{1}{a^2}+\frac{1}{b^2} \geq \frac{8}{t^2} \text{ (equality iff } a=b).$$

### 8 *Solution*

**(a)** IF $a>0$, $b>0$, $c>0$ and $a+b+c=1$, using the inequality

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\geq 9$$

(see Example 4), we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 9 \text{ (equality iff } a=b=c=\frac{1}{3}\text{)}.$$

Further, consider

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{(bc)^2+(ac)^2+(ab)^2}{a^2b^2c^2}.$$

By using the inequality $x^2+y^2+z^2\geq xy+yz+zx$ for $x>0$, $y>0$, $z>0$ (see Example 5 (**b**)) with respect to the numerator of the right-hand side of the last expression, we obtain

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq\frac{a^2bc+b^2ac+c^2ab}{a^2b^2c^2}=\frac{abc(a+b+c)}{a^2b^2c^2}=\frac{1}{abc}.$$

The inequality

$$\frac{(a+b+c)}{3}\geq\sqrt[3]{abc} \text{ (equality iff } a=b=c\text{)}$$

(see Example 5 (**b**)) in the case $a+b+c=1$ takes the form

$$\frac{1}{\sqrt[3]{abc}}\geq 3 \text{ or } \frac{1}{abc}\geq 27.$$

Hence

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 27 \text{ (equality iff } a=b=c=\frac{1}{3}\text{)}.$$

It was shown earlier that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 9 \text{ (equality iff } a=b=c=\frac{1}{3}\text{), or } \frac{ab+bc+ca}{abc}\geq 9.$$

In view of $abc\geq 0$, we obtain $ab+bc+ca\geq 9abc$ (equality iff $a=b=c=\frac{1}{3}$). Let us consider the expression

$$(1-a)(1-b)(1-c)-8abc=1-a-b-c+ab+bc+ca-abc-8abc=$$

$$=ab+bc+ca-9abc.$$

If we take into account the last inequality, we get

$$(1-a)(1-b)(1-c)\geq 8abc \text{ (equality iff } a=b=c=\frac{1}{3}\text{)}.$$

## 9 *Solution*

(a) Let $f(x)=e^x-1-x$. It is clear that $f'(x)=e^x-1>0$ for $x>0$. Thus, $f(x)$ is a not decreasing function for $x>0$. Function $f(x)$ has an absolute minimum of 0 when $x=0$. Hence, for $x>0$ $f(x)>0$, and $e^x>1+x$ for $x>0$.

(b) Show that $x>\dfrac{3\sin x}{2+\cos x}$ for $x>0$. Let us consider a function $f(x)=x-\dfrac{3\sin x}{2+\cos x}$. It is clear that

$$f'(x)=1-\frac{3\cos x(2+\cos x)+3\sin^2 x}{(2+\cos x)^2}=1-\frac{3+6\cos x}{(2+\cos x)^2}$$

$$=\frac{1+\cos^2 x-2\cos x}{(2+\cos x)^2}=\frac{(1-\cos x)^2}{(2+\cos x)^2}\ge 0.$$

Thus, $f(x)$ is a not decreasing function for $x>0$. Function $f(x)$ has an absolute minimum of 0 when $x=0$. Hence, for $x>0$ $f(x)>0$, and

$$x>\frac{3\sin x}{2+\cos x} \qquad \text{for } x>0.$$

## 10 *Solution*

(a) Let us prove that for $t>0$

$$1-t<\frac{1}{1+t}<1-t+t^2.$$

First, it is easily seen that $(1-t)(1+t)=1-t^2<1$ for $t>0$, and

$$1-t<\frac{1}{1+t} \quad \text{for } t>0.$$

Further, it is clear that $t^3=(1-t+t^2)(1+t)-1>0$ for $t>0$. Thus, we have

$$\frac{1}{1+t}<1-t+t^2 \quad \text{for } t>0.$$

Hence, we arrive to the desired result

$$1-t<\frac{1}{1+t}<1-t+t^2 \quad \text{for } t>0.$$

By integrating the last inequality between 0 and x, we derive

$$\int_0^x (1+t)dt<\int_0^x \frac{dt}{1+t}<\int_0^x (1-t+t^2)dt,$$

$$x-\frac{1}{2}x^2<\ln(1+x)<x-\frac{1}{2}x^2+\frac{1}{3}x^3 \qquad \text{for } x>0.$$

**(b)** Let us prove that for $t>0$

$$1-t^2<\frac{1}{1+t^2}<1-t^2+t^4.$$

First, it is easily seen that $(1-t^2)(1+t^2)=1-t^4<1$ for $t>0$, and

$$1-t^2<\frac{1}{1+t^2} \quad \text{for } t>0.$$

Further, it is clear that $t^6=(1-t^2+t^4)(1+t^2)-1>0$ for $t>0$. Thus, we have

$$\frac{1}{1+t^2}<1-t^2+t^4 \quad \text{for } t>0.$$

Hence, we arrive to the desired result

$$1-t^2<\frac{1}{1+t^2}<1-t^2+t^4 \quad \text{for } t>0.$$

By integrating the last inequality between 0 and x, we derive

$$\int_0^x (1-t^2)dt<\int_0^x \frac{dt}{1+t^2}<\int_0^x (1-t^2+t^4)dt,$$

$$x-\frac{1}{3}x^3<\tan^{-1}x<x-\frac{1}{3}x^3+\frac{1}{5}x^5 \qquad \text{for } x>0.$$

## 11 *Solution*

It is easily seen that for $0<t<1$ we have

$$\frac{1}{1+t}-\frac{1}{2}=\frac{1-t}{2(1+t)}>0, \quad \frac{1}{2}<\frac{1}{1+t},$$
$$1-\frac{1}{1+t}=\frac{t}{(1+t)}>0, \quad 1>\frac{1}{1+t}.$$

Hence,

$$\frac{1}{2}<\frac{1}{1+t}<1 \qquad \text{for } 0<t<1.$$

By integrating this inequality between 0 and u, we deduce that for $0<u<1$

$$\frac{1}{2}\int_0^u dt<\int_0^u \frac{1}{1+t}dt<\int_0^u dt,$$

$$\frac{u}{2}<\ln(1+u)<u.$$

**12 *Solution*.**

It is easily seen that for $t>0$ we have

$$\frac{1}{1+t}-\frac{1}{(1+t)^2}=\frac{t}{(1+t)^2}>0,\quad \frac{1}{(1+t)^2}<\frac{1}{1+t},$$
$$1-\frac{1}{1+t}=\frac{t}{(1+t)}>0,\quad 1>\frac{1}{1+t}.$$

Hence,

$$\frac{1}{(1+t)^2}<\frac{1}{1+t}<1 \qquad \text{for } t>0.$$

By integrating this inequality between 0 and u, we deduce that for $0<u<1$

$$\int_0^u \frac{1}{(1+t)^2}\,dt<\int_0^u \frac{1}{1+t}dt<\int_0^u dt,$$

$\frac{u}{1+u}<\ln(1+u)<u$.

# Exercise 8.2

### 1 *Solution*

Define the statement $S(n)$: $2\cdot 1!+5\cdot 2!+10\cdot 3!+\ldots+(n^2+1)n!=n(n+1)!$ for $n\geq 1$.

Consider $S(1)$: $n=1$, $2\cdot 1!=1\cdot 2!$. Hence $S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true, then

$$2\cdot 1!+5\cdot 2!+10\cdot 3!+\ldots+(k^2+1)k!=k(k+1)!.$$

Consider $S(k+1)$. If $S(k)$ is true, we get

$$2\cdot 1!+5\cdot 2!+10\cdot 3!+\ldots+(k^2+1)k!+((k+1)^2+1)(k+1)!=k(k+1)!+((k+1)^2+1)(k+1)!=$$

$$=(k+1)!(k+(k+1)^2+1)=(k+1)!(k+1)(k+2)=(k+2)!(k+1).$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n$:

$$2\cdot 1!+5\cdot 2!+10\cdot 3!+\ldots+(n^2+1)n!=n(n+1)! \quad \text{for} \quad n\geq 1.$$

### 2 *Solution*

Define the statement $S(n)$: $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots+\dfrac{n}{(n+1)!}=1-\dfrac{1}{(n+1)!}$ for $n\geq 1$.

Consider $S(1)$: $n=1$, $\dfrac{1}{2!}=1-\dfrac{1}{2!}=\dfrac{1}{2}$. Hence $S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true, then $\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots+\dfrac{k}{(k+1)!}=1-\dfrac{1}{(k+1)!}$.

Consider $S(k+1)$. If $S(k)$ is true, we get

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}=$$

$$=1-\frac{k+2-(k+1)}{(k+2)!}=1-\frac{1}{(k+2)!}.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n$:

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!} \quad \text{for} \quad n\geq 1.$$

### 3 *Solution*

Define the statement $S(n)$: $\quad u_n = 3\cdot 2^n + 1$ for $n \geq 1$.

Consider $S(1)$: $n=1$, $u_1 = 3\cdot 2+1 = 7 \Rightarrow S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true, then $u_k = 3\cdot 2^k + 1$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = 2u_k - 1 = 2\cdot(3\cdot 2^k + 1) - 1 = 3\cdot 2^{k+1} + 1.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = 3\cdot 2^n + 1 \quad \text{for} \quad n \geq 1.$$

### 4 *Solution*

Define the statement $S(n)$: $\quad u_n = 2 + 3^n$ for $n \geq 1$.

Consider $S(1)$: $n=1$, $u_1 = 2+3 = 5 \Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2$, $u_2 = 2+3^2 = 11 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer, $k \geq 2$. If $S(n)$ is true for all integer $n \leq k$, then

$$u_n = 2 + 3^n, \quad n = 1,2,3,\ldots,k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n = 1,2,3,\ldots,k$, we get

$$u_{k+1} = 4u_k - 3u_{k-2} = 4(2+3^k) - 3(2+3^{k-1}) = 2 + 4\cdot 3^k - 3\cdot 3^{k-1} =$$
$$= 2 + 3^{k+1}.$$

Hence for $k \geq 2$, $S(n)$ true for all positive integers $n \leq k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = 2 + 3^n \quad \text{for} \quad n \geq 1.$$

### 5 *Solution*

Define the statement $S(n)$: $\quad u_n = (n+3)2^n$ for $n \geq 1$.

Consider $S(1)$: $n=1$, $u_1 = 4\cdot 2 = 8 \Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2$, $u_2 = 5\cdot 4 = 20 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer, $k \geq 2$. If $S(n)$ is true for all integer $n \leq k$, then

$$u_n = (n+3)2^n, \quad n = 1,2,3,\ldots,k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n = 1,2,3,\ldots,k$, we get

$$u_{k+1} = 4u_k - 4u_{k-2} = 4(k+3)2^k - 4((k-1)+3)2^{k-1} = 4k2^k + 12\cdot 2^k - 4k2^{k-1} - 8\cdot 2^{k-1} =$$

$$= 2^{k+1}(k+4).$$

Hence for $k \geq 2$, $S(n)$ true for all positive integers $n \leq k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = (n+3)2^n \quad \text{for} \quad n \geq 1.$$

### 6 *Solution*

Define the statement $S(n)$: $\quad u_n = 2^n + 5^n \quad$ for $\quad n \geq 1$.

Consider $\quad S(1):\ n = 1, \quad u_1 = 2 + 5 = 7 \Rightarrow S(1)$ is true.

Consider $\quad S(2):\ n = 2, \quad u_2 = 2^2 + 5^2 = 29 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer, $k \geq 2$. If $S(n)$ is true for all integer $n \leq k$, then

$$u_n = 2^n + 5^n, \quad n = 1,2,3,\ldots,k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n = 1,2,3,\ldots,k$, we get

$$u_{k+1} = 7u_k - 10u_{k-2} = 7(2^k + 5^k) - 10(2^{k-1} + 5^{k-1}) =$$

$$= 7\cdot 2^k + 7\cdot 5^k - 5\cdot 2^k - 2\cdot 5^k = 2^{k+1} + 5^{k+1}.$$

Hence for $k \geq 2$, $S(n)$ true for all positive integers $n \leq k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = 2^n + 5^n \quad \text{for} \quad n \geq 1.$$

### 7 *Solution*

Define the statement $S(n)$: $\quad u_n = 2\cdot 3^n - 1 \quad$ for $\quad n \geq 1$.

Consider $\quad S(1):\ n = 1, \quad u_1 = 2\cdot 3 - 1 = 5 \Rightarrow S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true, then $u_k = 2\cdot 3^k - 1$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = 3u_k + 2 = 3\cdot(2\cdot 3^k - 1) + 2 = 2\cdot 3^{k+1} - 1.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = 2\cdot 3^n - 1 \quad \text{for} \quad n \geq 1.$$

### 8 *Solution*

Define the statement $S(n)$: $u_n = 5^n - 3$ for $n \geq 1$.

Consider $S(1)$: $n=1$, $u_1 = 5-3 = 2 \Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2$, $u_2 = 5^2 - 3 = 22 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer, $k \geq 2$. If $S(n)$ is true for all integer $n \leq k$, then

$$u_n = 5^n - 3, \quad n = 1,2,3,\mathrm{K}\,, k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n = 1,2,3,\mathrm{K}\,, k$, we get

$$u_{k+1} = 6u_k - 5u_{k-2} = 6(5^k - 3) - 5(5^{k-1} - 3) =$$
$$= 6\cdot 5^k - 5^k - 18 + 15 = 5^{k+1} - 3.$$

Hence for $k \geq 2$, $S(n)$ true for all positive integers $n \leq k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = 5^n - 3 \quad \text{for} \quad n \geq 1.$$

### 9 *Solution*

Define the statement $S(n)$: $u_n = (n-1)\cdot 3^n$ for $n \geq 1$.

Consider $S(1)$: $n=1$, $u_1 = 0 \Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2$, $u_2 = 1\cdot 3^2 = 9 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer, $k \geq 2$. If $S(n)$ is true for all integer $n \leq k$, then

$$u_n = (n-1)\cdot 3^n, \quad n = 1,2,3,\mathrm{K}\,, k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n = 1,2,3,\mathrm{K}\,, k$, we get

$$u_{k+1} = 6u_k - 9u_{k-2} = 6(k-1)\cdot 3^k - 9((k-1)-1)\cdot 3^{k-1} =$$
$$= k3^{k+1}.$$

Hence for $k \geq 2$, $S(n)$ true for all positive integers $n \leq k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = (n-1)\cdot 3^n \quad \text{for} \quad n \ge 1.$$

## 10 *Solution*

Define the statement $S(n)$: $\quad u_n = 5^n - 3^n$ for $n \ge 1$.

Consider $S(1)$: $n=1$, $u_1 = 5-3 = 2 \Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2$, $u_2 = 5^2 - 3^2 = 16 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer, $k \ge 2$. If $S(n)$ is true for all integer $n \le k$, then

$$u_n = 5^n - 3^n, \quad n = 1,2,3,\ldots,k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n = 1,2,3,\ldots,k$, we get

$$u_{k+1} = 8u_k - 15u_{k-2} = 8(5^k - 3^k) - 15(5^{k-1} - 3^{k-1}) =$$
$$= 8\cdot 5^k - 3\cdot 5^k - 8\cdot 3^k + 5\cdot 3^k = 5^{k+1} - 3^{k+1}.$$

Hence for $k \ge 2$, $S(n)$ true for all positive integers $n \le k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = 5^n - 3^n \quad \text{for} \quad n \ge 1.$$

## 11 *Solution*

It is easily seen that

$$u_{n+1} = 9^{n+2} - 8(n+1) - 9 = 9\cdot 9^{n+1} - 8n - 17 =$$
$$= 9(9^{n+1} - 8n - 9) + 72n + 81 - 8n - 17 = 9u_n + 64n + 64.$$

For $n \ge 1$ let the statement $S(n)$ be defined by: $\quad u_n$ is divisible by 64.

Consider $S(1)$: $n=1$, $u_1 = 64 \Rightarrow S(1)$ is true, since $u_1$ is divisible by 64.

Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k = 64\cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = 9u_k + 64k + 64 = 9\cdot 64M + 64k + 64 = 64(9M + k + 64).$$

Since $9M + k + 64$ is integer, we see that $u_{k+1}$ is divisible by 64. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n$ is divisible by 64 for $n \ge 1$.

### 12 *Solution*

It is easily seen that

$$u_{n+1} = 5^{2n+2} + 3(n+1) - 1 = 5^2 \cdot 5^{2n} + 3n + 2 =$$
$$= 25(5^{2n} + 3n - 1) - 75n + 25 + 3n + 2 = 25u_n - 72n + 27.$$

For $n \geq 1$ let the statement $S(n)$ be defined by: $u_n$ is divisible by 9.

Consider $S(1)$: $n = 1$, $u_1 = 5^2 + 3 - 1 = 27 \Rightarrow S(1)$ is true, since $u_1$ is divisible by 9.

Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k = 9 \cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = 25u_k - 72k + 27 = 25 \cdot 9M - 72k + 27 = 9(25M - 8k + 3).$$

Since $25M - 8k + 3$ is integer, we see that $u_{k+1}$ is divisible by 9. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n$ is divisible by 9 for $n \geq 1$.

### 13 *Solution*

It is easily seen that

$$u_{n+1} = 2^{n+3} + 3^{2n+3} = 2(2^{n+2} + 3^{2n+1}) - 2 \cdot 3^{2n+1} + 9 \cdot 3^{2n+1} =$$
$$= 2u_n + 7 \cdot 3^{2n+1}.$$

For $n \geq 1$ let the statement $S(n)$ be defined by: $u_n$ is divisible by 7.

Consider $S(1)$: $n = 1$, $u_1 = 8 + 27 = 35 = 7 \cdot 5 \Rightarrow S(1)$ is true, since $u_1$ is divisible by 7.

Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k = 7 \cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = 2u_k + 7 \cdot 3^{2k+1} = 2 \cdot 7M + 7 \cdot 3^{2k+1} = 7(2M + 3^{2k+1}).$$

Since $2M + 3^{2k+1}$ is integer, we see that $u_{k+1}$ is divisible by 7. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n$ is divisible by 7 for $n \geq 1$.

## 14 *Solution*

It is easily seen that

$$u_{n+1} = 3^{4(n+1)+2} + 2\cdot 4^{3(n+1)+1} = 3^{4n+6} + 2\cdot 4^{3n+4} =$$
$$= 3^4(3^{4n+2} + 2\cdot 4^{3n+1}) - 3^4\cdot 2\cdot 4^{3n+1} + 2\cdot 4^{3n+4} = 81u_n - 4^{3n+1}\cdot 34.$$

For $n \geq 1$ let the statement $S(n)$ be defined by: $u_n$ is divisible by 17.

Consider $S(1)$: $n = 1$, $u_1 = 3^6 + 2\cdot 4^4 = 1241 = 17\cdot 73 \Rightarrow S(1)$ is true, since $u_1$ is divisible by 17.

Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k = 17\cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = 81u_k - 4^{3k+1}\cdot 34 = 81\cdot 17\cdot M - 4^{3k+1}\cdot 2\cdot 17 = 17(81M - 2\cdot 4^{3k+1}).$$

Since $81M - 2\cdot 4^{3k+1}$ is integer, we see that $u_{k+1}$ is divisible by 17. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n$ is divisible by 17 for $n \geq 1$.

## 15 *Solution*

Let us introduce $f(n) = 7^n + 11^n$. It is easily seen that

$$f(n+2) = 7^{n+2} + 11^{n+2} = 7^2(7^n + 11^n) - 49\cdot 11^n + 121\cdot 11^n = 49f(n) + 72\cdot 11^n$$

For $n = 1,3,5,K$ let the statement $S(n)$ be defined by: $f(n)$ is divisible by 9 for odd $n \geq 1$.

Consider $S(1)$: $n = 1$, $f(1) = 7 + 11 = 18 = 9\cdot 2 \Rightarrow S(1)$ is true, since $f(1)$ is divisible by 9.

Let $k$ be a positive odd integer. If $S(k)$ is true for all integer $k$, then $f(k) = 9\cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$f(k+2) = 49f(k) + 11^k\cdot 72 = 49\cdot 9M + 11^k\cdot 8\cdot 9 = 9(49\cdot M + 11^k\cdot 8).$$

Since $49M + 8\cdot 11^k$ is integer, we see that $f(k+2)$ is divisible by 9. Hence for all odd positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all odd positive integers $n$: $7^n + 11^n$ is divisible by 9 for odd $n \geq 1$.

### 16 *Solution*

Let us introduce $f(n)=3^n+7^n$. It is easily seen that

$$f(n+2)=3^{n+2}+7^{n+2}=9(3^n+7^n)-9\cdot 7^n+49\cdot 7^n=9f(n)+40\cdot 7^n$$

For $n=1,3,5,K$ let the statement $S(n)$ be defined by: $f(n)$ is divisible by 10 for odd $n\geq 1$.

Consider $S(1)$: $n=1$, $f(1)=10\Rightarrow S(1)$ is true, since $f(1)$ is divisible by 10.

Let $k$ be a positive odd integer. If $S(k)$ is true for all integer $k$, then $f(k)=10\cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$f(k+2)=9f(k)+7^k\cdot 40=9\cdot 10M+7^k\cdot 4\cdot 10=10(9\cdot M+7^k\cdot 4).$$

Since $9M+4\cdot 7^k$ is integer, we see that $f(k+2)$ is divisible by 10. Hence for all odd positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all odd positive integers $n$: $3^n+7^n$ is divisible by 10 for odd $n\geq 1$.

### 17 *Solution*

It is easily seen that

$$\begin{aligned} u_{n+1}=3^{n+1}-2(n+1)-1=3(3^{n+1}-2n-1)+6n+3-2n-3= \\ =3u_n+4n. \end{aligned}$$

For $n\geq 2$ let the statement $S(n)$ be defined by: $u_n>0$ for $n\geq 2$.

Consider $S(2)$: $n=2$, $u_2=3^3-2\cdot 3-1=20>0\Rightarrow S(2)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k>0$ for $k\geq 2$. Consider $S(k+1)$, $k\geq 2$. If $S(k)$ is true, we get

$$u_{k+1}=3u_k+4\cdot k>0.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(2)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n=3^n-2n-1>0$ for $n\geq 2$.

### 18 *Solution*

It is easily seen that

$$u_{n+1} = 5^{n+1} - 4(n+1) - 1 = 5(5^n - 4n - 1) + 20n + 5 - 4n - 5 =$$
$$= 5u_n + 16n.$$

For $n \geq 2$ let the statement $S(n)$ be defined by: $u_n > 0$ for $n \geq 2$.

Consider $S(2)$: $n = 2$, $u_2 = 5^3 - 4 \cdot 3 - 1 = 112 > 0 \Rightarrow S(2)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k > 0$ for $k \geq 2$. Consider $S(k+1)$, $k \geq 2$. If $S(k)$ is true , we get

$$u_{k+1} = 5u_k + 16 \cdot k > 0.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(2)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n = 5^n - 4n - 1 > 0$ for $n \geq 2$.

### 19 *Solution*

**(a)** For $n = 1,2,3,K$ let the statement $S(n)$ be defined by: $u_n < 2$ for $n \geq 1$.

Consider $S(1)$: $n = 1$, $u_1 = 1 < 2 \Rightarrow S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k < 2$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true , we get

$$u_{k+1} = \sqrt{2u_k} = \sqrt{2}\sqrt{u_k} < \sqrt{2}\sqrt{2} = 2,$$

because of $\sqrt{u_k} < \sqrt{2}$, and $u_{k+1} < 2$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$: $u_n < 2$ for $n \geq 1$.

**(b)** For $n = 1,2,3,K$ let the statement $S(n)$ be defined by: $u_n < u_{n+1}$ for $n \geq 1$.

Consider $S(1)$: $n = 1$, $u_1 < u_2$, since $u_1 = 1, u_2 = \sqrt{2}$. Hence $S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k < u_{k+1}$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true , we get

$$u_{k+1} = \sqrt{2u_k} < \sqrt{2u_{k+1}} = u_{k+2},$$

because of $\sqrt{u_k} < \sqrt{u_{k+1}}$, and $u_{k+1} < u_{k+2}$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$: $u_n < u_{n+1}$ for $n \geq 1$.

## 20 *Solution*

**(a)** For $n = 1,2,3,K$ let the statement $S(n)$ be defined by: $u_n < 3$ for $n \geq 1$.

Consider $S(1)$: $n = 1$, $u_1 = 1 < 3 \Rightarrow S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k < 3$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = \sqrt{3+2u_k} < \sqrt{3+2\cdot 3} < 3,$$

because of $u_k < 3$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$: $u_n < 3$ for $n \geq 1$.

**(b)** For $n = 1,2,3,K$ let the statement $S(n)$ be defined by: $u_n < u_{n+1}$ for $n \geq 1$.

Consider $S(1)$: $n = 1$, $u_1 < u_2$, since $u_1 = 1, u_2 = \sqrt{5}$. Hence $S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k < u_{k+1}$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = \sqrt{3+2u_k} < \sqrt{3+2u_{k+1}} = u_{k+2},$$

because of $u_k < u_{k+1}$, and we see that $u_{k+1} < u_{k+2}$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$: $u_n < u_{n+1}$ for $n \geq 1$.

# Exercise 8.3

## 1 *Solution*

**(a)** In figure 8.1, ABC is a triangle. E and F are the midpoints of AC and AB respectively. BE and CF intersect at G. AG produced cuts BC at D. It is clear that BE and CF are two medians. Let us use the well-known theorem that three medians of a triangle intersect at a single point, which divides each median in accordance with the relation 2:1 starting from the top. This point is G, AD is the third median, and BD=CD. H is the point on AGD produced, such that AG=GH. Show that GBHC is a parallelogram. To this end we must prove that GB and CH, GC and BH are parallel to each other. According to the theorem mentioned above we get

$$\frac{AG}{GD}=\frac{2}{1},$$

and, if AG=GH, then $GD=\frac{1}{2}AG=\frac{1}{2}GH$.

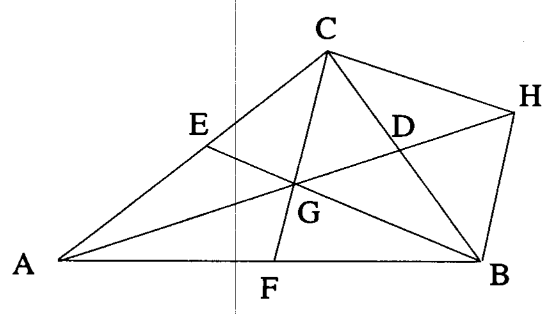


*Figure 8.1*

Hence $GD=DH$, since DH=GH-GD. It is easily seen that the triangles GBD and CDH coincide, since $GD=DH$, BD=CD, and $\angle GDB=\angle CDH$. Hence GB is parallel to CH.

The same goes for the triangles GDC and HDB. Thus CG is parallel to BH. Consequently GBHC is a parallelogram.

**(b)** We have showed above that BD=DC.

## 2 *Solution*

**(a)** In figure 8.2, ABC is a triangle. The internal bisectors of $\angle ABC$ and $\angle ACB$ meet at D. DP, DQ and DR are the perpendiculars from D to BC, AC and AB respectively. CD produced cuts AB at $C_1$, and BD produced cuts AC at $B_1$. $BB_1$ is the bisector of $\angle ABC$. This means that its points are equidistant with respect to the AB and BC, and $DR=DP$. The same goes for the points of the bisector $CC_1$, and $DQ=DP$. Hence $DR=DQ$.

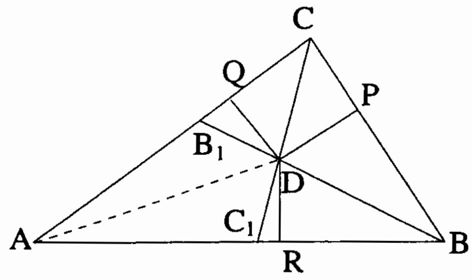


Figure 8.2

**(b)** As $DQ = DR$, the points of AD are equidistant with respect to sides AC and AB of the triangle ABC. Hence AD is the internal bisector of $\angle CAB$.

### 3 *Solution*

**(a)** In figure 8.3, AB is a chord of a circle. X is a point on AB. XT is a tangent from X to the circle. It is a well-known theorem that two triangles are similar if two angles of the first triangle are equal to two angles of the other one. Show that the triangles XAT and XTB are similar. To this end, let us remember the theorem about an inscribed angle that states: the inscribed angle $\angle$BAT is equal to the arch length BT measured in degrees divided by two. For the same reason $\angle$XTB is equal to the arch length BT measured in degrees divided by two. Hence $\angle$BAT =$\angle$XTB. The angle $\angle$TXA is common for both triangles XAT and XTB. This way $\Delta XAT$ and $\Delta XTB$ are similar.

**(b)** Since the triangles XAT and XTB are similar, it follows that the lengths of their sides which correspond to $\angle$TXA are proportional to each other, that is, $\frac{XA}{XT} = \frac{XT}{XB}$.

Hence $XA \cdot XB = XT^2$.

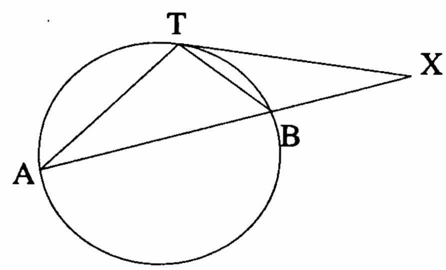


Figure 8.3

### 4 *Solution*

**(a)** In figure 8.4, AB and CD are chords of a circle. AB produced and CD produced meet at X. Show that $\Delta XAC$ and $\Delta XDB$ are similar. Let us use the well-known theorem that states: two triangles are similar if the corresponding angles are equal to each other. It is clear that $\angle$AXC = $\angle$BXD. $\angle$ACD = $\angle$ABD, since these two inscribed angles correspond to the same arch AD. Each of them equal the arch length AD measured in degrees divided by 2. Furthermore, $\angle$CAB = $\angle$CDB by the same reason. Hence the corresponding angles for both triangles are equal to each other, and $\Delta XAC$ and $\Delta XDB$ are similar.

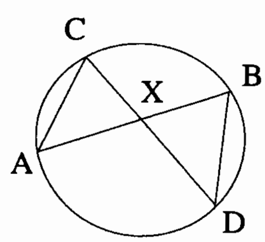


Figure 8.4

(**b**) Since $\Delta XAC$ and $\Delta XDB$ are similar, it follows that the lengths of the sides which correspond to the same angle are proportional to each other. Hence $\frac{XA}{XD} = \frac{XC}{XB} \Rightarrow XA \cdot XB = XC \cdot XD$.

**5 *Solution***

(**a**) In figure 8.5 a, ABD and AJK are two isosceles triangles with right angle at A. It is clear that the triangle AJK can be regarded as the result of rotation of the isosceles triangle $AJ_1K_1$ with angle $\varphi$ with respect to the center A. Hence the triangles ABJ and ADK coincide, and $\angle BJA = \angle DKA$.

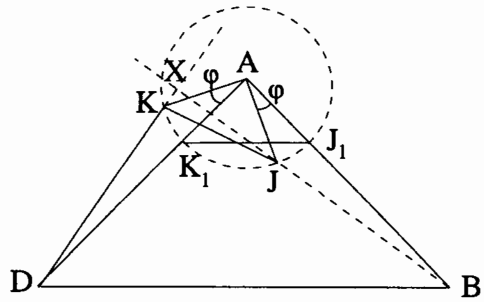


*Figure 8.5 a*

(**b**) BJ is produced to meet DK at X. The triangles ABJ and ADK coincide. Since $AJ \perp AK$ and $AB \perp AD$, we get $BJ \perp DK$. Hence $BX \perp DK$.

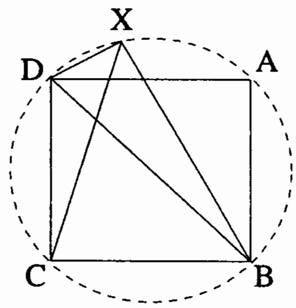


*Figure 8.5 b*

(**c**) In figure 8.5 b, the square ABCD is completed. The angle $\angle DXB$ is $90°$. Let us consider a circle based on the points A, B, C, D. It is clear that X belongs to the circle ($\angle DXB$ and $\angle DAB$ are two inscribed angles based on the arch DB and equal to $90°$). According to the theorem about an inscribed angle $\angle BXC$ is equal to the arch length BC measured in degrees divided by $2$, that is, $\angle BXC = 90°/2 = 45°$. Hence XC is the bisector of $\angle DXB$.

**6 *Solution***

(**a**) In figure 8.6, Two circles with centres O and P and radii r and s (where $r < s$) respectively touch externally at T. ABC and ADE are common tangents to the circles. It is clear that $\angle ABO$, $\angle ADO$ and $\angle ACP$, $\angle AEP$ are right angles, and $OD = OB = r$, $PE = PC = s$. Hence AO (or AP) is the bisector of $\angle CAE$, and the points O and P lie on this line. The point T also belongs to the bisector AP, since T lies on the line OP; A, O, T and P are collinear.

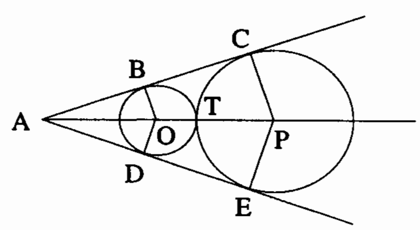


*Figure 8.6*

**(b)** It is easily seen that $\Delta\,AOB$ and $\Delta\,APC$ are similar triangles. Hence

$$\frac{OB}{AO}=\frac{PC}{AP}\Rightarrow\frac{r}{AO}=\frac{s}{AO+r+s}\Rightarrow$$
$$(AO+r+s)\cdot r=s\cdot AO\Rightarrow AO=\frac{r(r+r)}{s-r}.$$

### 7 *Solution*

In $\Delta\,ABC$, $AB=AC$ (see figure 8.7). The bisector of $\angle ABC$ meets AC at K. The circle through A, B and K cuts BC at D. Let $\angle ABC = 2\alpha$. The inscribed angles $\angle ABK$ and $\angle KBC$ are based on the arches AK and DK being equal to each other as $\angle ABK$ and $\angle KBC = \alpha$. (the inscribed angle is equal to the arch length, it corresponds, measured in degrees divided by 2). Hence $AK=DK$. Then the inscribed angles $\angle KBD$ and $\angle DAK$ are equal to each other as they based on the common arch DK. Thus we get $\angle DAK = \alpha$. The triangle AKD is isosceles since $AK=DK$. As a result we get $\angle AKD = 180\text{-}2\alpha$ thus $\angle DKC = 2\alpha = \angle ACB$. Hence the triangle CKD is isosceles and $DK=CD$. Finally, we come to the desired result $AK=CD$.

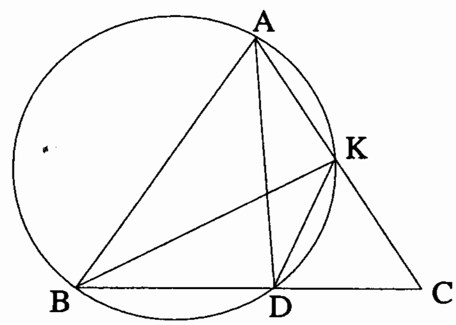

*Figure 8.7*

### 8 *Solution*

In triangle ABC, P and Q are points on the sides CA and AB respectively, such that $\angle BPC = \angle CQB$ (see figure 8.8). BP and CQ intersect at K. X and Y are points on CA and AB respectively, such that AYKX is a parallelogram. It is clear that $\angle QBK = \angle KCP$ as $\angle QKB = \angle PKC$ and $\angle BPC = \angle CQB$. Then we get arrive at $\angle BYK = \angle KXC$, since the lines BY, YK and KX, XC are parallel to each other.

Hence the triangles BKY and CKX are similar as they have two angles which are equal to each other. As a result their sides are proportional

$$\frac{YK}{XK} = \frac{YB}{XC}.$$

As $AY = XK$ and $YK = AX$, we get

$$\frac{AX}{AY} = \frac{YB}{XC} \Rightarrow AX \cdot XC = AY \cdot YB.$$

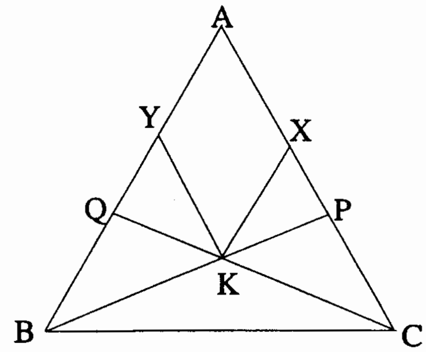


*Figure 8.8*

**9 *Solution***

PQ and RS are two chords of a circle (see Figure 8.9 a). PQ and RS intersect at H. K is a point such ∠KPQ and ∠KRS are right angles. Let ∠RKH = α and ∠PKH = β, ∠HRP = y and RPH = x. The points K,P,H and R lie on the circle with diameter KH, as the angles ∠KRH and ∠KPH are right (see Figure 8.9 b). According to the theorem of the inscribed angle we get that $x = \alpha,\ y = \beta$, since ∠RPH and ∠RKH are based on the common arch RH. The same holds for ∠HRP and ∠PKH. On Figure 8.9 a we see that ∠PQS and ∠PRS are based on the arch PS, and ∠RSQ and ∠RPQ are based on the arch RQ. Hence ∠PQS = ∠PRS = y = β, ∠RSQ = ∠RPQ = x = α. Since KR is perpendicular to RS and ∠RKH = RSQ = α, we find that KH is perpendicular to QS. Otherwise this contradicts the statement of the equal angles between perpendicular lines.

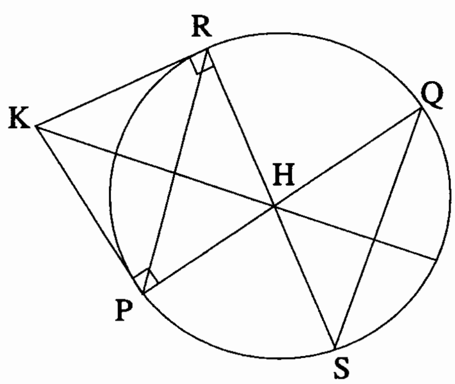


*Figure 8.9 a*

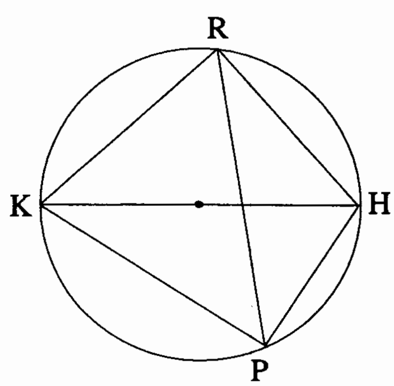


*Figure 8.9 b*

## 10 *Solution*

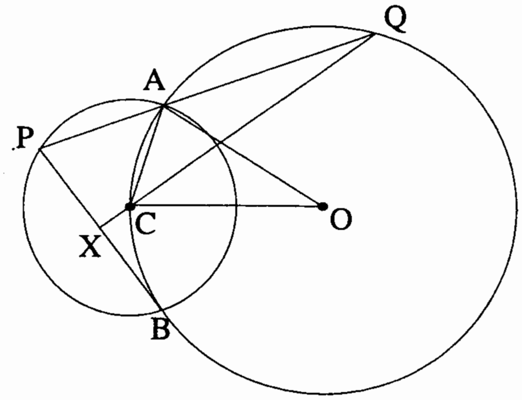


*Figure 8.10*

In figure 8.10, two circles intersect at A and B. The center C of the first circle lies on the second circle with center O. P is a point on the first circle and Q is a point on the second circle such that PAQ is a straight line. QC produced meets PB at X. Let $\angle AOC = \alpha$ and $\angle ACO = \beta$. It is clear that $\Delta AOC$ is isosceles, and $2\beta + \alpha = 180° \Rightarrow \beta + \alpha/2 = 90°$. $\angle AQC$ and $\angle AOC$ are based on the common arch AC. According to the theorem of the inscribed angle we get $\angle AQC = \alpha/2$. $\angle APB$ is based on the arch AB and being equal to $2\beta$ if measured in degrees. Hence $\angle APB = \beta$. Thus the sum of the angles $\angle PQX$ and $\angle QPX$ is equal to $\alpha/2 + \beta = 90°$. Hence $\angle PXQ = 90°$, and QX is perpendicular to PB.

# Diagnostic test 8

## 1 *Solution*

It is clear that

$$a^2+b^2\geq 2ab.$$

Hence, if $a>0$ and $b>0$, multiplication this inequality on $a$ and $b$ yields

$$a^3+ab^2\geq 2a^2b,$$

$$a^2b+b^3\geq 2ab^2.$$

By addition of these inequalities we come to

$$a^3+b^3+a^2b+ab^2\geq 2a^2b+2ab^2.$$

Hence

$$a^3+b^3\geq a^2b+ab^2 \text{ (equality iff } a=b\text{)}.$$

## 2 *Solution*

Consider

$$(a^2+b^2)(c^2+d^2)-(ac+bd)^2=a^2c^2+a^2d^2+b^2c^2+b^2d^2-a^2c^2-2abcd-b^2d^2=$$

$$=a^2d^2-2abcd+b^2c^2=(ad-bc)^2\geq 0.$$

Hence $(ac+bd)^2\leq(a^2+b^2)(c^2+d^2)$ with equality iff $ad=bc$.

**(a)** Consider $a^2+b^2\geq 2ab$. By adding $a^2+b^2$ into both sides of the inequality, we get

$$(a+b)^2\leq 2(a^2+b^2) \text{ (equality iff } a=b\text{)}.$$

**(b)** It is clear that

$$(a^2+b^2)(a^4+b^4)-(a^3+b^3)^2=a^2b^4-2a^3b^3+b^2a^4=a^2b^2(a^2-2ab+b^2)=$$

$$=a^2b^2(a-b)^2\geq 0.$$

Hence

$$(a^3+b^3)^2\leq(a^2+b^2)(a^4+b^4) \text{ (equality iff } a=b\text{)}.$$

## 3 *Solution*

It is clear that $(a+b)^2\geq 4ab$ (equality iff $a=b$), since $(a+b)^2=(a-b)^2+4ab$. If $a>0$ and $b>0$, we have

$$a+b \geq 2\sqrt{ab} \quad \text{(equality iff } a=b\text{)},$$

$$\frac{a+b}{2} \geq \sqrt{ab} \quad \text{(equality iff } a=b\text{)}.$$

**(a)** Consider

$$\frac{a+b+c+d}{4} = \frac{\dfrac{a+b}{2}+\dfrac{c+d}{2}}{2}.$$

Using the inequality proved above with respect to $\frac{a+b}{2}$ and $\frac{c+d}{2}$, we come to

$$\frac{a+b+c+d}{4} \geq \frac{\sqrt{ab}+\sqrt{cd}}{2}.$$

Employing the same inequality once again with respect to the right-hand side of the last inequality, we obtain

$$\frac{a+b+c+d}{4} \geq \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt[4]{abcd}.$$

Hence $a+b+c+d \geq 4\sqrt[4]{(abcd)}$ (equality iff $a=b=c=d$).

**(b)** After the substitution $a \to \frac{a}{b}$, $b \to \frac{b}{c}$, $c \to \frac{c}{d}$, $d \to \frac{d}{a}$ the last inequality becomes

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{d}{a} \geq 4 \quad \text{(equality iff } a=b=c=d\text{)}.$$

### 4 *Solution*

It is easily seen that for $0<t<1$

$$\frac{1}{1+t^2}-\frac{1}{2}=\frac{1-t^2}{2(1+t^2)}>0 \Rightarrow \quad \frac{1}{2}<\frac{1}{1+t^2}, \quad 1-\frac{1}{1+t^2}=\frac{t^2}{1+t^2}>0 \Rightarrow \quad \frac{1}{1+t^2}<1.$$

Hence $\frac{1}{2}<\frac{1}{1+t^2}<1$ for $0<t<1$. By integrating between 0 and $u$, we deduce

$$\int_0^u \frac{1}{2}dt < \int_0^u \frac{dt}{1+t^2} < \int_0^u dt, \quad \frac{1}{2}u < \ln(1+u) < u$$

for $0<u<1$.

### 5 *Solution*

Consider S(1): $n=1$ $\quad 1\cdot 1!=1=2!-1$, hence $S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true, then $1\cdot 1!+2\cdot 2!+\cdots+k\cdot k!=(k+1)!-1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+k\cdot k!+(k+1)(k+1)!=(k+1)!-1+(k+1)(k+1)!$$
$$=(k+1)!(1+k+1)-1=(k+1)!(k+2)-1=(k+2)!-1.$$

Hence for all positive $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$:

$$1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!=(n+1)!-1,\ n\geq 1.$$

### 6 *Solution*

Define the statement $S(n)$: $\quad u_n=3^n-2^n$ for $\quad n\geq 1$. Consider

$$S(1):\ n=1,\quad u_1=3^1-2^1=1\Rightarrow S(1) \text{ is true.}$$

Consider

$$S(2):\ n=2,\quad u_2=3^2-2^2=5\Rightarrow S(2) \text{ is true.}$$

Let $k$ be a positive integer, $k\geq 2$. If $S(n)$ is true for all integer $n\leq k$, then

$$u_n=3^n-2^n,\quad n=1,2,3,\ldots,k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n=1,2,3,\ldots,k$, we get

$$u_{k+1}=5u_k-6u_{k-1}=5(3^k-2^k)-6(3^{k-1}-2^{k-1})=53^k-2\cdot 3^k-5\cdot 2^k+3\cdot 2^k$$
$$=3\cdot 3^k-2\cdot 2^k=3^{k+1}-2^{k+1}.$$

Hence for $k\geq 2$, $S(n)$ true for all positive integers $n\leq k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers $n$:

$$u_n=3^n-2^n \quad \text{for} \quad n\geq 1.$$

### 7 *Solution*

It is easily seen that

$$u_{n+1}=5^{n+1}+12(n+1)-1=5\cdot(5^{n+1}+12n-1)-60n+5+12n+11$$
$$=5u_n-48n+16.$$

For $n\geq 1$ let the statement $S(n)$ be defined by: $\quad u_n$ is divisible by 16.

Consider $S(1): n=1, \quad u_1 = 16 \Rightarrow S(1)$ is true, since $u_1$ is divisible by 16.

Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k = 16 \cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = 5u_k - 48k + 16 = 5 \cdot 16M - 3 \cdot 16k + 16 = 16(5M - 3k + 1).$$

Since $5M - 3k + 1$ is integer, we see that $u_{k+1}$ is divisible by 16. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n$ is divisible by 16 for $n \geq 1$.

## 8 *Solution*

**(a)** For $n = 1,2,3,K$ let the statement $S(n)$ be defined by: $u_n < 2$ for $n \geq 1$.

Consider $S(1): n=1, \quad u_1 = 1 < 2 \Rightarrow S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k < 2$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = \sqrt{2 + u_k} < \sqrt{2+2} = 2,$$

because of $\sqrt{u_k} < \sqrt{2}$, and $u_{k+1} < 2$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$: $u_n < 2$ for $n \geq 1$.

**(b)** For $n = 1, 2, 3, K$ let the statement $S(n)$ be defined by: $u_n < u_{n+1}$ for $n \geq 1$.

Consider $S(1): n=1, \quad u_2 > u$, since $u_1 = 1, \; u_2 = \sqrt{3}$. Hence $S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_{k+1} > u_k$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = \sqrt{2 + u_k} < \sqrt{2 + u_{k+1}} = u_{k+2},$$

because of $2 + u_{k+1} > 2 + u_k$, and we get $u_{k+1} < u_{k+2}$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$: $u_n < u_{n+1}$ for $n \geq 1$.

### 9 *Solution*

**(a)** In figure 8.11, ABC is an acute-angled triangle. The altitudes BE and CF intersect at G. AG produced cuts BC at D. Let it be $\angle EAG = \alpha$, $\angle FAG = \beta$. It is a well known fact that a quadrilateral is cyclic if the sum of its opposite angles is equal to $180°$. Furthermore, the sum of the internal angles of a quadrilateral are equal to $360°$. As $\angle AEG = \angle AFG = 90°$, $\angle AEG + \angle AFG = 180°$, and $\alpha + \beta + \angle EGF = 180°$.

Hence AFGE is a cyclic quadrilateral. One can prove (see solution 9 in the exercise 8.3) that $\angle EFG = \alpha$ and $\angle FEG = \beta$. Let us use the theorem that states: three altitudes of a triangle intersect at a single point. Hence AD is the third altitude, and $\angle ADB = \angle ADC = 90°$. This enables us to arrive at $\angle GBD = \alpha$ and $\angle GCD = \beta$. Then we get

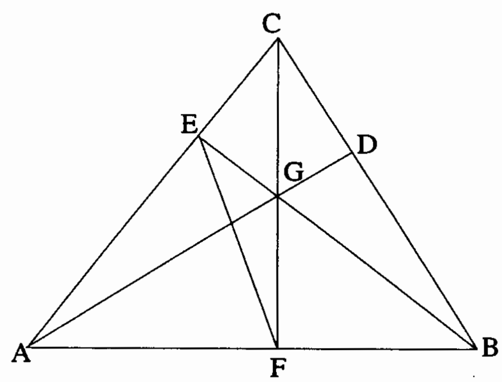


*Figure 8.11*

$\angle EFB + \angle ECD = \alpha + 90° + \beta + \angle ECG = 180°$,

since $\angle ECG = 90o - \alpha - \beta$.

In an analogous way we obtain $\angle FEC + \angle FBD = 180°$. Hence CEFB is a cyclic quadrilateral.

**(b)** It is clear that $\angle FGA = 90° - b$ and $\angle FBD = a + \angle FBG = \alpha + 90° - \alpha - \beta = 90° - \beta$. Hence $\angle FGA = \angle FBD$. Furthermore, AD is perpendicular to BC as AD is the altitude.

### 10 *Solution*

In figure 8.12, ABC is an acute-angled triangle. The altitudes AD and BE intersect at G. AD produced cuts the circle through A, B and C at H. The inscribed angles $\angle BHA$ and $\angle BCA$ are based on the arch AB. According to the theorem of the inscribed angle $\angle BHA = \angle BCA$. The sum of the internal angles of the quadrilateral CDGE is equal to $360°$. Since $\angle GEC = \angle GDC = 90°$, we get $\angle ECD + \angle DGE = 180°$. Furthermore, $\angle BGH = 180° - \angle AGB = 180°$

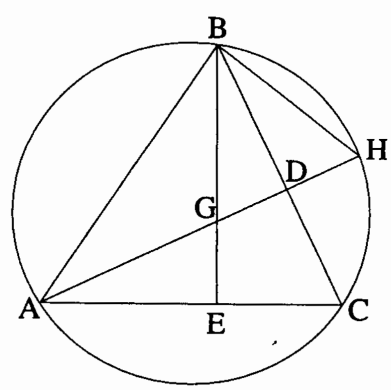


*Figure 8.12*

- $\angle DGE = \angle ECD$. Hence $\angle BGH = \angle BHA$, as $\angle ECD = \angle BCA$, and $\Delta BHG$ is the isosceles triangle with the altitude BD. As a result we get $GD = DH$.

# Further questions 8

### 1 *Solution*

Let $a=(l,m,n)$ and $b=(x,y,z)$ be vectors in a three dimension coordinate space with scalar product $(a,b)=l\,x+m\,y+n\,z$. Furthermore, we have

$$(a,a)=l^2+m^2+n^2\geq 0,$$
$$(b,b)=x^2+y^2+z^2\geq 0.$$

It is clear that for real $\lambda$ we get

$$0\leq(\lambda\,a-b,\lambda\,a-b)=(\lambda\,l-x)^2+(\lambda\,m-y)^2+(\lambda\,n-z)^2=\lambda^2(a,a)-2\lambda(a,b)+(b,b).$$

In the right-hand side the polynomial of second order with respect to $\lambda$ is not negative. Hence, $(a,b)^2-(a,a)(b,b)$ must be negative or equal to zero, and, we come to

$$(a,b)^2\leq(a,a)\cdot(b,b),$$

that is,

$$(lx+my+nz)^2\leq(l^2+m^2+n^2)(x^2+y^2+z^2).$$

The equality takes place, if for any real $t$ the following relations hold: $l=t\,x,\quad m=t\,y,\quad n=t\,z$.

**(a)** Consider

$$3(a^2+b^2+c^2)-(a+b+c)^2=2(a^2+b^2+c^2-ab-bc-ac)$$
$$=(a-b)^2+(a-c)^2+(b-c)^2\geq 0.$$

Hence $3(a^2+b^2+c^2)\geq(a+b+c)^2$ ( equality iff $a=b=c$).

**(b)** Let us consider an expression

$$(a^2+b^2+c^2)(a^4+b^4+c^4)-(a^3+b^3+c^3)^2=a^6+a^2b^4+a^2c^4+b^2a^4+b^6+$$
$$+b^2c^4+c^2a^4+c^2b^4+c^6-a^6-b^6-c^6-2a^3b^3-2a^3c^3-2b^3c^3$$
$$=a^2b^2(b^2+a^2)+a^2c^2(c^2+a^2)+b^2c^2(c^2+b^2)-2a^3b^3-2a^3c^3-2b^3c^3.$$

By using $2ab\leq a^2+b^2$, $2ac\leq a^2+c^2$ and $2bc\leq b^2+c^2$, we get

$$2a^3b^3+2a^3c^3+2b^3c^3\leq a^2b^2(a^2+b^2)+a^2c^2(a^2+c^2)+b^2c^2(b^2+c^2).$$

In view of this inequality we come to the final result

$$(a^2+b^2+c^2)(a^4+b^4+c^4)\geq(a^3+b^3+c^3)^2 \text{ ( equality iff } a=b=c \text{ ).}$$

## 2 Solution

Let $a>0, b>0, c>0$ and $d>0$. First consider the inequality

$$(A+B+C)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\geq 9$$

with positive $A, B, C$ (see example 4 in the Exercise 8.1).

It is clear that

$$\frac{9}{a+b+c}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \qquad \frac{9}{a+c+d}\leq\frac{1}{a}+\frac{1}{c}+\frac{1}{d},$$

$$\frac{9}{a+b+d}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{d}, \qquad \frac{9}{b+c+d}\leq\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

By addition

$$\frac{3}{b+c+d}+\frac{3}{a+b+c}+\frac{3}{a+b+d}+\frac{3}{a+c+d}\leq\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

( equality iff $a=b=c=d$ ).

Furthermore, if we employ the inequality $a+\frac{1}{a}\geq 2,\quad a>0$ (see example 4 in the Exercise 8.1), we have

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=1+1+1+1+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{d}+\frac{d}{b}\right)+$$

$$\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{d}+\frac{d}{c}\right)+\left(\frac{d}{a}+\frac{a}{d}\right)\geq 4+2+2+2+2+2+2=16.$$

Hence $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 16$ ( equality iff $a=b=c=d$ ).

The substitution $a\to a+b+c,\quad b\to b+c+d,\quad c\to c+d+a,\quad d\to d+a+b$ in this inequality permits to obtain the desired factor $\frac{1}{b+c+d}+\frac{1}{a+b+c}+\frac{1}{a+b+d}+\frac{1}{a+c+d}$, and we get

$$3(a+b+c+d)\left(\frac{1}{a+b+c}+\frac{1}{a+b+d}+\frac{1}{b+c+d}+\frac{1}{a+c+d}\right)\geq 16,$$

( equality iff $a=b=c=d$ ).

Then

$$\frac{16}{a+b+c+d} \le \frac{3}{b+c+d} + \frac{3}{a+b+c} + \frac{3}{a+b+d} + \frac{3}{a+c+d}.$$

Finally, using the inequalities given above

$$\frac{3}{b+c+d} \le \frac{1}{3}\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right), \qquad \frac{3}{a+b+c} \le \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

$$\frac{3}{a+b+d} \le \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}\right), \qquad \frac{3}{a+c+d} \le \frac{1}{3}\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{d}\right),$$

we come to the desired result

$$\frac{16}{a+b+c+d} \le \frac{3}{b+c+d} + \frac{3}{a+b+c} + \frac{3}{a+b+d} + \frac{3}{a+c+d} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

( equality iff $a=b=c=d$ ).

### 3 *Solution*

Show by differentiation that $xy \le e^{x-1} + y\ln y$ for all real $x$ and all positive $y$. When does equality hold?

Let $f(x) = e^{x-1} + y\ln y - xy$ be the function with parameter $y > 0$. It is easily to get

$f'(x) = e^{x-1} - y$, $f'(x) = 0 \Leftrightarrow e^{x-1} = y$ or $x = 1 + \ln y$.

Furthermore, $f'(x) > 0$ for $x > 1 + \ln y$ and $f'(x) < 0$ for $x < 1 + \ln y$, as we can see that, if

$x = \Delta x + 1 + \ln y$, then $f'(x) = y(e^{\Delta x} - 1)$, and

$f'(x) > 0$ if $\Delta x > 0$, and $f'(x) < 0$ if $\Delta x < 0$.

Hence $f(x)$ has an absolute minimum of $0$ when $x = 1 + \ln y$. As a result, we get for all $x$

$f(x) \ge 0$, and $e^{x-1} + y\ln y \ge xy$ with equality iff $x = 1 + \ln y$.

### 4 *Solution*

**(a)** Let us evaluate the following integrals:

$$\int_0^1 x^2(1-x)^2 dx = \int_0^1 x^2(1-2x+x^2)dx = \int_0^1 x^2 dx - 2\int_0^1 x^3 dx + \int_0^1 x^4 dx = \left[\frac{x^3}{3}\right]_0^1 - 2\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1$$

$$=\frac{1}{3}-\frac{2}{4}+\frac{1}{5}=\frac{1}{30}.$$

$$\int_0^1 \frac{x^2(1-x)^2}{x+2}dx=\int_0^1 \frac{x^2-2x^3+x^4}{x+2}dx.$$

By using $x^4-2x^3+x^2=(x+2)(x^3-4x^2+9x-18)+36$, we get

$$\int_0^1 \frac{x^2(1-x)^2}{x+2}dx=\int_0^1 (x^3-4x^2+9x-18)dx+36\int_0^1 \frac{dx}{x+2}$$

$$=\left[\frac{x^4}{4}\right]_0^1-4\left[\frac{x^3}{3}\right]_0^1+9\left[\frac{x^2}{2}\right]_0^1-18[x]_0^1+36[\ln(x+2)]_0^1=\frac{1}{4}-\frac{4}{3}+\frac{9}{2}-18+36\ln\frac{3}{2}$$

$$=-\frac{175}{12}+36\ln\frac{3}{2}.$$

**(b)** It is easily seen that for $0<x<1$

$$\frac{1}{3}<\frac{1}{x+2}<\frac{1}{2},$$

because

$$\frac{1}{x+2}-\frac{1}{3}=\frac{1-x}{x+2}>0,$$
$$\frac{1}{2}-\frac{1}{x+2}=\frac{x}{x+2}>0.$$

Since $x^2(1-x)^2>0$, we get $\frac{1}{3}x^2(1-x)^2<\frac{x^2(1-x)^2}{x+2}<\frac{1}{2}x^2(1-x)^2$. By integrating this inequality with respect to $x$ between $0$ and $1$, we deduce that

$$\frac{1}{3}\int_0^1 x^2(1-x)^2dx<\int_0^1 \frac{x^2(1-x)^2}{x+2}dx<\frac{1}{2}\int_0^1 x^2(1-x)^2dx.$$

In view of $\int_0^1 x^2(1-x)^2dx=\frac{1}{30}$, $\int_0^1 \frac{x^2(1-x)^2}{x+2}dx=36\ln\frac{3}{2}-\frac{175}{12}$ (see Solution 4(a)), we obtain

$$\frac{1}{90}<36\ln\frac{3}{2}-\frac{175}{12}<\frac{1}{60}\Rightarrow\frac{2630}{180}<36\ln\frac{3}{2}<\frac{876}{60}\Rightarrow\frac{2627}{6480}<\ln\frac{3}{2}<\frac{2628}{6480}.$$

**5 Solution**

**(a)**

$$\int_0^1 x^4(1-x)^4dx = \int_0^1 (x^4-4x^5+6x^6-4x^7+x^8)dx = \left[\frac{x^5}{5}\right]_0^1 - 4\left[\frac{x^6}{6}\right]_0^1 + 6\left[\frac{x^7}{7}\right]_0^1 - 4\left[\frac{x^8}{8}\right]_0^1 + \left[\frac{x^9}{9}\right]_0^1$$

$$= \frac{1}{5} - \frac{2}{3} + \frac{6}{7} - \frac{1}{2} + \frac{1}{9} = \frac{1}{630}.$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \frac{x^4-4x^5+6x^6-4x^7+x^8}{1+x^2}dx$$

By using the representation

$$x^8-4x^7+6x^6-4x^5+x^4 = (1+x^2)(x^6-4x^5+5x^4-4x^2+4)-4,$$

we get

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 (x^6-4x^5+5x^4-4x^2+4)dx - 4\int_0^1 \frac{dx}{1+x^2}$$

$$= \left[\frac{x^7}{7}\right]_0^1 - 4\left[\frac{x^6}{6}\right]_0^1 + 5\left[\frac{x^5}{5}\right]_0^1 - 4\left[\frac{x^3}{3}\right]_0^1 + 4[x]_0^1 - 4\left[\tan^{-1}x\right]_0^1 = \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi.$$

**(b)** It is easily seen that for $0 < x < 1$

$$\frac{1}{2} < \frac{1}{1+x^2} < 1,$$

because of $\dfrac{1}{1+x^2} - \dfrac{1}{2} = \dfrac{1-x^2}{2(1+x^2)} > 0, \quad 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0.$

Since $x^4(1-x)^4 > 0$, we get $\dfrac{1}{2}x^4(1-x)^4 < \dfrac{x^4(1-x)^4}{1+x^2} < x^4(1-x)^4$.

By integrating this inequality with respect to $x$ between $0$ and $1$, we deduce that

$$\frac{1}{2}\int_0^1 x^4(1-x)^4dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx < \int_0^1 x^4(1-x)^4dx.$$

In view of

$$\int_0^1 x^4(1-x)^4dx = \frac{1}{630},$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7} - \pi$$

(see Solution 5(a)), we obtain

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630} \Rightarrow -\frac{1}{630} < \pi - \frac{22}{7} < -\frac{1}{1260} \Rightarrow \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

### 6 *Solution*

Let us show that $\sin x < x$ for $0 < x < \dfrac{\pi}{2}$. It is easily to deduce that for $f(x) = x - \sin x$, we get $f'(x) = 1 - \cos x \geq 0$.

Hence for $x \geq 0$ $f(x)$ is a non-decreasing function with absolute minimum $0$ when $x = 0$.

Thus $f(x) > 0$ for $x > 0$, and $\sin x < x$ for $0 < x < \dfrac{\pi}{2}$.

Let us show that $\sin x > \dfrac{2}{\pi}x$ for $0 < x < \dfrac{\pi}{2}$. It is not difficult to establish for

$$g(x) = \sin x - \frac{2}{\pi}x$$

that $g'(x) = \cos x - \dfrac{2}{\pi}$ and $g'(x) = 0$ when $x = \arccos\dfrac{2}{\pi}$. Furthermore, for $0 < x < \dfrac{\pi}{2}$ function $g(x)$ has the only absolute maximum of $\sin(\arccos 2/\pi) - 2/\pi \arccos 2/\pi > 0$ when $x = \arccos 2/\pi$, since

$$g'(x) = \cos x - \frac{2}{\pi} > 0 \text{ for } x < \arccos\frac{2}{\pi},$$
$$g'(x) = \cos x - \frac{2}{\pi} < 0 \text{ for } x > \arccos\frac{2}{\pi}.$$

Function $g(x)$ reaches absolute minimum of $0$ when $x = 0, \dfrac{\pi}{2}$.

Thus $g(x) \geq 0$ for $0 < x < \dfrac{\pi}{2}$, that is, $\dfrac{2}{\pi}x < \sin x$, and, finally,

$$\frac{2}{\pi}x < \sin x < x,$$
$$-\frac{2}{\pi}x > -\sin x > -x,$$
$$e^{-x\cdot 2/\pi} > e^{-\sin x} > e^{-x}.$$

By integrating the last inequality with respect to $x$ between $0$ and $\dfrac{\pi}{2}$, we come to

$$\int_0^{\pi/2} e^{-x}dx < \int_0^{\pi/2} e^{-\sin x}dx < \int_0^{\pi/2} e^{-x\cdot 2/\pi}dx\,,\quad -\left[e^{-x}\right]_0^{\pi/2} < \int_0^{\pi/2} e^{-\sin x}dx < -\frac{\pi}{2}\left[e^{-x\cdot 2/\pi}\right]_0^{\pi/2},$$

$$1-e^{-\pi/2} < \int_0^{\pi/2} e^{-\sin x}dx < \frac{\pi}{2e}(e-1)\,.$$

### 7 *Solution*

Define the statement $S(n)$: $1\cdot\ln\frac{2}{1}+2\cdot\ln\frac{3}{2}+\ldots+n\cdot\ln\left(\frac{n+1}{n}\right)=\ln\left(\frac{(n+1)^n}{n!}\right)$ for $n\geq 1$.

Consider $S(1)$: $n=1$, $1\cdot\ln\frac{2}{1}=\ln\left(\frac{2}{1}\right)=\ln 2$. Hence $S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true, then

$$1\cdot\ln\frac{2}{1}+2\cdot\ln\frac{3}{2}+\ldots+k\cdot\ln\left(\frac{k+1}{k}\right)=\ln\left(\frac{(k+1)^k}{k!}\right).$$

Consider $S(k+1)$. If $S(k)$ is true, we get

$$1\cdot\ln\frac{2}{1}+2\cdot\ln\frac{3}{2}+\ldots+k\cdot\ln\left(\frac{k+1}{k}\right)+(k+1)\cdot\ln\left(\frac{k+2}{k+1}\right)=\ln\left(\frac{(k+1)^k}{k!}\right)+(k+1)\cdot\ln\left(\frac{k+2}{k+1}\right)$$

$$=\ln\left[\frac{(k+1)^k}{k!}\cdot\left(\frac{k+2}{k+1}\right)^{k+1}\right]=\ln\left[\frac{(k+2)^{k+1}}{(k+1)!}\right].$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n$:

$$1\cdot\ln\frac{2}{1}+2\cdot\ln\frac{3}{2}+\ldots+n\cdot\ln\left(\frac{n+1}{n}\right)=\ln\left(\frac{(n+1)^n}{n!}\right).$$

### 8 *Solution*

Define the statement $S(n)$:

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\ldots+\frac{x(x+1)\ldots(x+n-1)}{n!}=\frac{(x+1)(x+2)\ldots(x+n)}{n!}\quad\text{for}\quad n\geq 1.$$

Consider $S(1)$: $n=1$, $1+\frac{x}{1!}=1+x=\frac{x+1}{1}$. Hence $S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true, then

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\cdots+\frac{x(x+1)\cdots(x+k-1)}{k!}=\frac{(x+1)(x+2)\cdots(x+k)}{k!}.$$

Consider $S(k+1)$. If $S(k)$ is true, we get

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\cdots+\frac{x(x+1)\cdots(x+k-1)}{k!}+\frac{x(x+1)\cdots(x+k)}{(k+1)!}$$
$$=\frac{(x+1)(x+2)\cdots(x+k)}{k!}+\frac{x(x+1)\cdots(x+k)}{(k+1)!}=\frac{(x+1)(x+2)\cdots(x+k)(k+1+x)}{(k+1)!}.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n$:

$$1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\cdots+\frac{x(x+1)\cdots(x+n-1)}{n!}=\frac{(x+1)(x+2)\cdots(x+n)}{n!}.$$

## 9 *Solution*

Let $u_n$ be given by $u_n = 35^n + 3\cdot 7^n + 2\cdot 5^n + 6$.

It is easily seen that

$$\begin{aligned}u_{n+1} &= 35^{n+1}+3\cdot 7^{n+1}+2\cdot 5^{n+1}+6\\ &= 35(35^n+3\cdot 7^n+2\cdot 5^n+6)-105\cdot 7^n-70\cdot 5^n-210+21\cdot 7^n+10\cdot 5^n+6\\ &= 35\cdot u_n-84\cdot 7^n-60\cdot 5^n-204.\end{aligned}$$

For $n \geq 1$ let the statement $S(n)$ be defined by: $u_n$ is divisible by 12.

Consider $S(1)$: $n=1$, $u_1 = 35+21+10+6 = 12\cdot 6 \Rightarrow S(1)$ is true, since $u_1$ is divisible by 12.

Let $k$ be a positive integer. If $S(k)$ is true for all integer $k$, then $u_k = 12\cdot M$ for some integer $M$. Consider $S(k+1)$. If $S(k)$ is true , we get

$$\begin{aligned}u_{k+1} &= 35\cdot u_k-84\cdot 7^k-60\cdot 5^k-204 = 35\cdot 12M-12\cdot 7\cdot 7^k-12\cdot 5\cdot 5^k-17\cdot 12\\ &= 12\cdot(35M-7^{k+1}-5^{k+1}-17).\end{aligned}$$

Since $35M-7^{k+1}-5^{k+1}-17$ is integer, we see that $u_{k+1}$ is divisible by 12. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all positive integers $n$: $u_n$ is divisible by 12 for $n \geq 1$.

### 10 *Solution*

**(a)** Let $P_n(x)$ be a polynomial of degree $n$ and given by $P_n(x)=(1+x)^n-1$. It is clear that

$$P_{n+1}(x)=(1+x)^{n+1}-1=(1+x)\cdot(1+x)^n-1=(1+x)\big((1+x)^n-1\big)+1+x-1$$
$$=P_n(x)\cdot(1+x)+x.$$

Define the statement $S(n): P_n(x)$ is divisible by $x$ for $n\geq 1$.

Consider $S(1): P_1(x)=1+x-1=x$ is divisible by $x \Rightarrow S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true, then $P_k(x)=x\cdot R_{k-1}(x)$ for a polynomial

$R_{k-1}(x)$

of degree $k-1, k\geq 1$. Consider $S(k+1)$:

$$P_{k+1}(x)=P_k(x)\cdot(1+x)+x=x\cdot R_{k-1}(x)(1+x)+x=x\cdot(R_{k-1}(x)(1+x)+1).$$

Since $R_{k-1}(x)\cdot(1+x)+1$ is a polynomial of degree $k$ $(k\geq 1)$, $P_{k+1}(x)$ is divisible by $x$. Hence for all positive integers $k, S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true.

Hence by induction, $S(n)$ is true for all positive integers $n\geq 1$: $P_n(x)=(1+x)^n-1$ is divisible by $x$ for $n\geq 1$.

**(b)** Let $Q_n(x)$ be a polynomial of degree $n$ and given by

$Q_n(x)=(1+x)^n-1-nx;\quad n\geq 2$. It is clear that

$$Q_{n+1}(x)=(1+x)^{n+1}-(n+1)x-1=(1+x)((1+x)^n-nx-1)+(1+x)nx+1+x-(n+1)x-1$$
$$=(1+x)\cdot Q_n(x)+x^2n.$$

Define the statement $S(n)$: $Q_n(x)$ is divisible by $x^2$ for $n\geq 2$.

Consider $S(2)$: $Q_2(x)=(1+x)^2-1-2x=x^2$ is divisible by $x^2 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer and $k\geq 2$. If $S(k)$ is true, then $Q_k(x)=x^2\cdot R_{k-2}(x)$ for a polynomial $R_{k-2}(x)$ of degree $k-2,\ k\geq 2$. Consider $S(k+1)$:

$$Q_{k+1}(x)=(1+x)Q_k(x)+x^2k=(1+x)x^2R_{k-2}(x)+x^2k=x^2\cdot((1+x)\cdot R_{k-2}(x)+k).$$

Since $R_{k-2}(x)\cdot(1+x)+k$ is a polynomial of degree $k-1$ $(k\geq 2)$, $Q_{k+1}(x)$ is divisible by $x^2$. Hence for all positive integers $k, S(k)$ true implies $S(k+1)$ true. But $S(2)$ is true.

Hence by induction, $S(n)$ is true for all positive integers $n \geq 1$: $Q_n(x) = (1+x)^n - 1 - nx$ is divisible by $x^2$ for $n \geq 2$.

## 11 *Solution*

**(a)** Define the statement $S(n)$: $\frac{d}{dx}x^n = n \cdot x^{n-1}, \quad n \geq 1$.

Consider $S(1)$: $\frac{d}{dx}x = 1 \cdot x^0 = 1 \Rightarrow S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true then $\frac{d}{dx}x^k = k \cdot x^{k-1}, \quad k \geq 1$.

Consider $S(k+1)$. If $S(k)$ is true, we get by using the product rule for differentiation

$$\frac{d}{dx}x^{k+1} = \frac{d}{dx}(x \cdot x^k) = x^k \frac{d}{dx} \cdot x + x \cdot \frac{d}{dx}x^k = x^k + x \cdot k \cdot x^{k-1} = x^k \cdot (k+1), \quad k \geq 1.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n \geq 1$: $\frac{d}{dx}x^n = n \cdot x^{n-1}$.

**(b)** Define the statement $S(n)$: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \geq 1$.

Consider $S(1)$: $\int x dx = \frac{x^2}{2} + c \quad \Rightarrow \quad S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true then $S(k)$: $\int x^k dx = \frac{x^{k+1}}{k+1} + c$ for $k \geq 1$.

Consider $S(k+1)$. If $S(k)$ is true, we get $x^k dx = \frac{dx^{k+1}}{k+1}$. Using integration by parts leads to

$$I = \int x^{k+1} dx = \int x \cdot x^k dx = \int x \cdot \frac{dx^{k+1}}{k+1} = \frac{x^{k+2}}{k+1} - \frac{1}{k+1}\int x^{k+1} dx = \frac{x^{k+2} - I}{k+1}.$$

Hence

$$I \cdot \left(1 + \frac{1}{k+1}\right) = \frac{x^{k+2}}{k+1} \quad \Rightarrow \quad I = \frac{x^{k+2}}{k+2}.$$

Finally, we get $\int x^{k+1} dx = \frac{x^{k+2}}{k+2} + c$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \geq 1.$$

### 12 *Solution*

**(a)** Define the statement $S(n)$: $\dfrac{d^n}{dx^n}\ln(1+x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$, $n \geq 1$.

Consider $S(1)$: $\dfrac{d}{dx}\ln(1+x) = \dfrac{0!}{(1+x)^1} = \dfrac{1}{1+x} \Rightarrow S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true then $\dfrac{d^k}{dx^k}\ln(1+x) = \dfrac{(-1)^{k-1}\cdot(k-1)!}{(1+x)^k}$, $k \geq 1$.

Consider $S(k+1)$. If $S(k)$ is true, we get by using the product rule for differentiation

$$\frac{d^{k+1}}{dx^{k+1}}\ln(1+x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\ln(1+x)\right) = \frac{d}{dx}\cdot\frac{(-1)^{k-1}\cdot(k-1)!}{(1+x)^k} = (-1)^{k-1}(k-1)!\cdot\frac{d}{dx}\frac{1}{(1+x)^k}$$

$$= \frac{(-1)^k(k-1)!\, k}{(1+x)^{k+1}} = \frac{(-1)^k \cdot k!}{(1+x)^{k+1}}, \quad k \geq 1.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers

$n \geq 1$: $\dfrac{d^n}{dx^n}\ln(1+x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$.

**(b)** Define the statement $S(n)$: $\dfrac{d^n}{dx^n}\ln(1-x) = -\dfrac{(n-1)!}{(1-x)^n}$, $n \geq 1$.

Consider $S(1)$: $\dfrac{d}{dx}\ln(1-x) = -\dfrac{1}{1-x} \Rightarrow S(1)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true then $\dfrac{d^k}{dx^k}\ln(1-x) = -\dfrac{(k-1)!}{(1-x)^k}$, $k \geq 1$.

Consider $S(k+1)$. If $S(k)$ is true, we get by using the product rule for differentiation

$$S(k+1): \quad \frac{d^{k+1}}{dx^{k+1}}\ln(1-x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\ln(1-x)\right) = \frac{d}{dx}\left(-\frac{(k-1)!}{(1-x)^k}\right) = -(k-1)!\frac{d}{dx}\left(\frac{1}{(1-x)^k}\right)$$

$$= -(k-1)!\, k\frac{1}{(1-x)^{k+1}} = -\frac{k!}{(1-x)^{k+1}}.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers

$$n \geq 1: \quad \frac{d^n}{dx^n}\ln(1-x) = -\frac{(n-1)!}{(1-x)^n}.$$

## 13 *Solution*

Let the function $f(n)$ define the quantity of diagonals for a convex polygon with $n \geq 4$ sides. It is easily seen that $f(n+1) = f(n) + n - 1$ (see figure 15), since including an additional point $A_{n+1}$ for a polygon with $n$ sides leads to new $n-2$ diagonals with respect to the points

$A_1$, $A_3$,K , $A_{n-1}$, besides the side $A_1A_n$ becomes a new diagonal.

Define the statement

$$S(n): \; f(n) = \frac{n(n-3)}{2} \quad \text{for } n \geq 4.$$

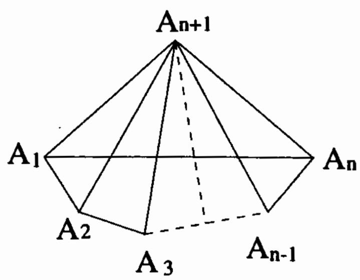

*Figure 15*

Consider $S(4)$: $\quad f(4) = \dfrac{4 \cdot 1}{2} = 2 \Rightarrow S(1)$ true.

Let $k$ be a positive integer, $k \geq 4$. If $S(k)$ is true for all integers $k \geq 4$, then $f(k) = \dfrac{k(k-3)}{2}$.

Consider $S(k+1)$. If $S(k)$ is true, we get

$$f(k+1) = f(k) + k - 1 = \frac{k(k-3)}{2} + k - 1 = \frac{k^2 - k - 2}{2} = \frac{(k+1)(k-2)}{2}.$$

We see that $S(k)$ true implies $S(k+1)$ true for $k \geq 4$. But $S(4)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n \geq 4$.

## 14 *Solution*

Let $u_n$ be the number of intersection points formed by $n \geq 2$ lines. We are seeking a recurrence relation between $u_{n+1}$ and $u_n$. The $(n+1)$th line intersects each of the other lines (see figure 16).

Hence we have $n$ distinct intersection points along the additional line, and

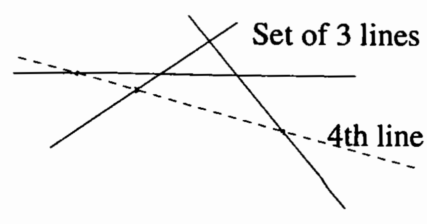

*Figure 16*

$$u_{n+1} = u_n + n, \quad n \geq 2.$$

Define the statement $S(n)$: $u_n = \dfrac{n(n-1)}{2}$, $n \geq 2$.

Clearly $S(2)$ is true, since two different lines give one intersection point. Let $k$ be a positive integer, $k \geq 2$. If $S(k)$ is true, then $u_k = \dfrac{k(k-1)}{2}$. Consider $S(k+1)$

$$u_{k+1} = u_k + k = \frac{k(k-1)}{2} + k = \frac{k^2 - k + 2k}{2} = \frac{k(k+1)}{2}.$$

Hence $u_{k+1} = \dfrac{k(k+1)}{2}$, if $S(k)$ is true.

Thus for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, hence $S(n)$ is true for all positive integers $n$: $n$ such lines have $\dfrac{n(n-1)}{2}$ points of intersection.

**15 *Solution***

Let $u_1 = 1$ and $u_n = \dfrac{2u_{n-1}^3 + 27}{3u_{n-1}^2}$ for $n \geq 2$.

**(a)** Define the statement $S(n)$: $u_n > 3$, $n \geq 2$.

Consider $S(2)$: $u_2 = \dfrac{2+27}{3 \cdot 1} = \dfrac{29}{3} > 3 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true then $u_k > 3$, $k \geq 2$.

Consider $S(k+1)$. Show that $u_{k+1} > 3$. To this end, let us consider a function $f(x)$ given by

$$f(x) = \frac{2}{3}x + \frac{9}{x^2}, x \geq 3.$$

It is easily seen that

$$f'(x) = \frac{2}{3} - \frac{18}{x^3}.$$

We obtain that $f'(x) = 0$ when $x = 3$, $f''(3) > 0$. Thus the function $f(x)$ has an absolute minimum of $f(3) = 3$. Hence

$$f(x) = \frac{2}{3}x + \frac{9}{x^2} > 3 \quad \text{for } x \geq 3.$$

Thus, if $S(k)$ is true ($u_k > 3, \quad k \geq 2$), using this inequality, we get

$$S(k+1): \quad u_{k+1} = \frac{2u_k^3 + 27}{3u_k^2} = \frac{2}{3}u_k + \frac{9}{u_k^2} > 3.$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(2)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n \geq 2$: $\quad u_n > 3$.

**(b)** Show that $u_{n+1} < u_n$ for $n \geq 2$.

One can deduce that $u_n - u_{n+1} = \frac{1}{3}u_n - \frac{9}{{u_n}^2}$, since $u_{n+1} = \frac{2}{3}u_n + \frac{9}{u_n^2}$.

Let the function $g(x)$ be given by $g(x) = \frac{1}{3}x - \frac{9}{x^2}, \quad x \geq 3$.

It is easy to see that $g'(x) = \frac{1}{3} + \frac{18}{x^3} > 0$ for $x \geq 3$. Hence $g(x)$ is a monotonously increasing function for $x \geq 3$ and $g(3) = 0$. Thus $g(x) > 0$ for $x > 3$, and

$\frac{1}{3}x - \frac{9}{x^2} > 0, \quad x > 3$.

By using this inequality and the fact proved in (a) that $u_n > 3, n \geq 2$, we get $\frac{1}{3}u_n - \frac{9}{{u_n}^2} > 0$.

Hence $u_{n+1} < u_n, n \geq 2$.

## 16 *Solution*

Let $u_1 = 1$ and $u_n = \frac{1}{2}\left(u_{n-1} + \frac{3}{u_{n-1}}\right)$ for $n \geq 2$.

**(a)** Define the statement $S(n)$: $\quad u_n^2 = \frac{1}{4}\left(u_{n-1}^2 + 6 + \frac{9}{u_{n-1}^2}\right) > 3$ for $n \geq 2$.

Consider $S(2)$: $\quad u_2^2 = \frac{1}{4}(1+6+9) = 4 > 3 \Rightarrow S(2)$ is true.

Let $k$ be a positive integer. If $S(k)$ is true then ${u_k}^2 > 3, \quad k \geq 2$ or

$$S(k): \quad u_k^2 = \frac{1}{4}\left(u_{k-1}^2 + 6 + \frac{9}{u_{k-1}^2}\right) > 3 \text{ for } k \geq 2.$$

Consider $S(k+1)$. Show that ${u_{k+1}}^2 > 3$, that is,

$$S(k+1): \quad u_{k+1}^2 = \frac{1}{4}\left(u_k^2 + 6 + \frac{9}{u_k^2}\right) > 3.$$

Show that $S(k+1)$ is true. Let the function $f(x)$ be given by

$$f(x) = \frac{1}{4}\left(x + 6 + \frac{9}{x}\right), \quad x \geq 3.$$

It is easy to see that $f'(x) = \frac{1}{4}\left(1 - \frac{9}{x^2}\right)$, and $f'(x) = 0$ when $x = 3$, $f''(3) > 0$.

We obtain that the function $f(x)$ has an absolute minimum of 3 when $x = 3$. Hence

$$\frac{1}{4}\left(x + 6 + \frac{9}{x}\right) > 3 \text{ for } x > 3.$$

By using this inequality, in view of $u_k^2 > 3$, we get $u_{k+1}^2 = \frac{1}{4}\left(u_k^2 + 6 + \frac{9}{u_k^2}\right) > 3$.

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(2)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n \geq 2$: $u_n{}^2 > 3$.

**(b)** Show that $u_{n+1} < u_n$ for $n \geq 2$.

One can deduce that $u_n - u_{n+1} = \frac{1}{2}u_n - \frac{3}{2u_n}$ and $u_n > 0$ for $n \geq 2$.

Let the function $g(x)$ be given by $g(x) = \frac{1}{2}x - \frac{3}{2x}$, $x > 0$.

It is easy to see that $g'(x) = \frac{1}{2} + \frac{3}{2x^2} > 0$ for all $x$.

Hence $g(x)$ is a monotonously increasing function and $g(x) = 0$ when $x = \sqrt{3}$. Thus $g(x) > 0$ for $x > \sqrt{3}$, and $\frac{1}{2}x - \frac{3}{2x} > 0$ for $x > \sqrt{3}$.

By using this inequality and the fact $u_n^2 > 3$ or $u_n > \sqrt{3}$, proved in (a), we get $\frac{1}{2}u_n - \frac{3}{2u_n} > 0$ for $n \geq 2$. Hence $u_{n+1} < u_n$ for $n \geq 2$.

## *17 Solution*

Show that for $n \geq 1$, $n! \geq 2^{n-1}$.

Define the statement $S(n)$: $n! \geq 2^{n-1}, n \geq 1$.

Consider $S(1)$: $n=1,\ 1!\geq 2^0=1$. Hence $S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true, then $k!\geq 2^{k-1}, k\geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get $(k+1)!=k!(k+1)\geq 2^{k-1}(k+1)=2^k\cdot\dfrac{k+1}{2}\geq 2^k$, since $\dfrac{(k+1)}{2}\geq 1,\quad k\geq 1$. Hence $(k+1)!\geq 2^k,\quad k\geq 1$. Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers $n$: $n!\geq 2^{n-1},\quad n\geq 1$.

Deduce that the statement $\dfrac{1}{(1!)^2}+\dfrac{1}{(2!)^2}+\cdots+\dfrac{1}{(n!)^2}\leq\dfrac{4}{3}\left(1-\dfrac{1}{4^n}\right),\quad n\geq 1$.

Consider $S(1)$: $n=1$, $\dfrac{1}{(1!)^2}\leq\dfrac{4}{3}\left(1-\dfrac{1}{4}\right)=1$.

Hence $S(1)$ is true. Let $k$ be a positive integer. If $S(k)$ is true, then

$$\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\cdots+\frac{1}{(k!)^2}\leq\frac{4}{3}\left(1-\frac{1}{4^k}\right).$$

Consider $S(k+1)$. If $S(k)$ is true, we get

$$S(k+1):\ \frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\cdots+\frac{1}{(k!)^2}+\frac{1}{((k+1)!)^2}\leq\frac{4}{3}\left(1-\frac{1}{4^k}\right)+\frac{1}{((k+1)!)^2}.$$

Since $k!\geq 2^{k-1},\quad k\geq 1$ we get $\dfrac{1}{((k+1)!)^2}\leq\dfrac{1}{4^k}$. By using this inequality, we come to

$$\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\cdots+\frac{1}{(k!)^2}+\frac{1}{((k+1)!)^2}\leq\frac{4}{3}\left(1-\frac{1}{4^k}\right)+\frac{1}{4^k}=\frac{4}{3}\left(1-\frac{1}{4^{k+1}}\right).$$

Hence for all positive integers $k$, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive $n$:

$$\frac{1}{(1!)^2}+\frac{1}{(2!)^2}+\cdots+\frac{1}{(n!)^2}\leq\frac{4}{3}\left(1-\frac{1}{4^n}\right).$$

### 18 *Solution*

Let $u_1=1, u_2=1$ and $u_n=u_{n-1}+u_{n-2}$ for $n\geq 3$.

Define the statement $S(n)$: $u_n=\dfrac{1}{\sqrt{5}}\left\{\left(\dfrac{1+\sqrt{5}}{2}\right)^n-\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right\}$ for $n\geq 1$.

Consider $S(1)$: $u_1 = \frac{1}{\sqrt{5}}\left\{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right\} = 1 \Rightarrow S(1)$ is true.

Consider $S(2)$: $u_2 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right\} = \frac{1}{\sqrt{5}}\left\{\frac{4\sqrt{5}}{4}\right\} = 1 \Rightarrow \mathrm{S}(2)$ is true.

Let $k$ be a positive integer, $k \geq 2$. If $S(n)$ is true for all integers $n \leq k$, then

$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ for $n = 1, 2, \ldots, k$. Consider $S(k+1)$:

$$u_{k+1} = u_k + u_{k-1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1+\sqrt{5}}{2} + 1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1-\sqrt{5}}{2} + 1\right)\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^2\right\} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\},$$

if $S(n)$ is true for $n = 1, 2, \ldots k$. For $k = 2, 3, \ldots$, $S(n)$ is true for all positive integers $n \leq k$ implies $S(k+1)$ is true. But $S(1), S(2)$ are true. Hence by induction, $S(n)$ is true for all positive integers $n$:

$$u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

## 19 *Solution*

ABCD is a quadrilateral such that ∠ABD = ∠DBC = ∠CDA = 45°. Q is the point on BD such that CQ bisects ∠BCA. It is easily seen that AQ is the bisector of ∠BAC as we know that the bisectors of Δ ABC intersect at the common point Q.

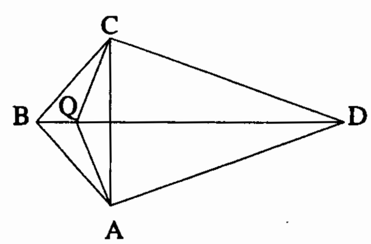


*Figure 8.13*

Since ∠ABC = 90°, we get

2∠QAC + 2.∠QCA + 90° = 180° thus ∠QAC + ∠QCA = 45°.

It is clear that $\angle QAC + \angle QCA + \angle AQC = 180^{o}$ thus $\angle AQC = 135^{o}$.

Hence $\angle AQC + \angle ADC = 135^{o} + 45^{o} = 180$. A quadrilateral is cyclic if the sums of its opposite angles are equal to $180^{o}$. Thus this is true for the angles $\angle AQC$ and $\angle ADC$. But we know that the sum of the internal angles of a convex quadrilateral is equal to $360^{0}$. Therefore we obtain $\angle QAD + \angle QCD = 360^{o}-(\angle AQC + \angle ADC) = 360^{o}-180^{o} = 180^{o}$. Hence the quadrilateral AQCD is a cyclic one.

### 20 *Solution*

ABC is a triangle. The bisector of $\angle CAB$ cuts BC at D. K is the point on CB produced such that $BK = AC$. AB produced cuts the circle through A, K and D at P. One can prove using the theorem of sines that the bisector AD produced cuts BC at D in such a way that the following relation holds: $\dfrac{BD}{AB} = \dfrac{DC}{AC}$. Hence

$$DC = AC \cdot BD / AB.$$

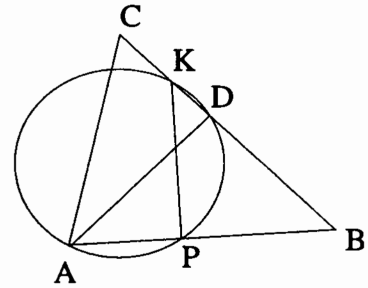


*Figure 8.14*

Then consider the triangles ABD and PBK with the common angle $\angle ABC$. Moreover, we see that $\angle PAD = \angle PKD$ as these inscribed angles are based on the common arch DP (theorem of inscribed angle). The triangles ABD and PBK are similar ( their two angles coincide). Hence their sides are proportional to each other: $\dfrac{PB}{BD} = \dfrac{BK}{AB}$. Using $AC = BK$, we get

$$PB = BD \cdot BK / AB = BD \cdot AC / AB.$$

If one compares the results derived for DC and PB, then it is clear that $BP = DC$.