7SD Solutions Series

Worked Solutions to Popular Mathematics Texts

Suggested Worked Solutions to

"4 Unit Mathematics"

(Text book for the NSW HSC by D. Arnold and G. Arnold)

Chapter 4
Polynomials

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Solutions are to “4 Unit Mathematics”

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Exercise 4.1

1 *Solution*

(a) (i) In order to solve the polynomial equation $P(x) = x^4 - 5x^2 + 4 = 0$, denote x^2 as t, then $t^2 - 5t + 4 = 0 \Rightarrow t_1 = 4, t_2 = 1$. Hence $t^2 - 5t + 4 = (t-4)(t-1)$ and thus $x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1) = (x-2)(x+2)(x-1)(x+1)$ are irreducible factors of $P(x)$ over **Q**. Each linear factor gives rise to a zero of $P(x)$. Hence the zeros of $P(x)$ are $\pm 1, \pm 2$. All these zeros are rational.

(i, ii) Over **R** and **C** this polynomial has the same zeros.

(b) Denoting in $P(x) = x^4 - 3x^2 + 2 = 0$, x^2 as t we obtain $t^2 - 3t + 2 = (t-2)(t-1)$, and thus $x^4 - 3x^2 + 2 = (x^2 - 2)(x^2 - 1) = (x^2 - 2)(x-1)(x+1)$ are irreducible factors of $P(x)$ over **Q**.

Hence the zeros of $P(x)$ over **Q** are ± 1.

(ii, iii) Irreducible factors of $P(x)$ over **R** and also **C** are

$P(x) = (x - \sqrt{2})(x + \sqrt{2})(x-1)(x+1)$.

Hence the zeros of $P(x)$ over **R** and **C** are $\pm 1, \pm\sqrt{2}$.

(c) (i, ii) Irreducible factors of $P(x)$ over **Q** and **R** are

$P(x) = (x^2 - 1)(x^2 + 4) = (x-1)(x+1)(x^2 + 4)$.

Hence the zeros of $P(x)$ over **Q** and **R** are ± 1.

(iii) $P(x) = (x-1)(x+1)(x-2i)(x+2i)$.

Hence the zeros of $P(x)$ over **C** are $\pm 1, \pm 2i$.

2 *Solution*

(a) (i) $P(x) = x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$, and these factors are irreducible over **Q**. Hence $P(x) = 0$ has no roots over **Q**.

(ii, iii) $P(x) = (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{3})(x + \sqrt{3})$. Hence the roots of $P(x) = 0$ over **R** and also **C** are $\pm\sqrt{2}, \pm\sqrt{3}$.

(b) (i) $P(x)=x^4-x^2-2=(x^2-2)(x^2+1) \Rightarrow$ the equation $P(x)=0$ has no roots over **Q**.

(ii) $P(x)=(x-\sqrt{2})(x+\sqrt{2})(x^2+1) \Rightarrow$ the roots over **R** are $\pm\sqrt{2}$.

(iii) $P(x)=(x-\sqrt{2})(x+\sqrt{2})(x-i)(x+i) \Rightarrow$ the roots over **C** are $\pm\sqrt{2}, \pm i$.

(c) (i, ii) $P(x)=(x^2+1)(x^2+4)$ and cannot be factored further over **Q** and **R**

$\Rightarrow P(x)=0$

has no roots over **Q** and **R**.

(iii) Irreducible factors of $P(x)$ over **C** are $P(x)=(x-i)(x+i)(x-2i)(x+2i)$.

Hence

the roots of $P(x)=0$ over **C** are $\pm i, \pm 2i$.

3 *Solution*

(a)

$$\begin{array}{r l}
 & x^2+1 \qquad \leftarrow \text{quotient} \\
x-1 \big) & \overline{x^3-x^2+x-1} \\
 & \underline{x^3-x^2} \\
 & \qquad\quad x-1 \\
 & \qquad\quad \underline{x-1} \\
 & \qquad\qquad 0 \leftarrow \text{remainder}
\end{array}$$

$\Rightarrow (x^3-x^2+x-1)=(x-1)(x^2+1)$. Also

$P(x)=x^3-x^2+x-1 \Rightarrow P(1)=1-1+1-1=0$.

(b)

$$\begin{array}{r l}
 & x^2+(i-1)\,x-i \qquad \leftarrow \text{quotient} \\
x-i \big) & \overline{x^3-x^2+x-1} \\
 & \underline{x^3-ix^2} \\
 & (i-1)\,x^2+ \quad x \quad -1 \\
 & \underline{(i-1)\,x^2+(1+i)\,x} \\
 & \qquad\qquad -ix-1 \\
 & \qquad\qquad \underline{-ix-1}
\end{array}$$

$$0 \leftarrow \text{remainder}$$

$\Rightarrow x^3 - x^2 + x - 1 = (x-i)\{x^2 + (i-1)\,x - i\}$. Also $P(i) = i^3 - i^2 + i - 1 = 0$.

4 *Solution*

(a)

$$\begin{array}{r l}
 & x^2 - 4x + 8 \quad \leftarrow \text{quotient} \\
x+1 \,\big) & x^3 - 3x^2 + 4x - 2 \\
 & \underline{x^3 + x^2} \\
 & -4x^2 + 4x - 2 \\
 & \underline{-4x^2 - 4x} \\
 & 8x - 2 \\
 & \underline{8x + 8} \\
 & -10 \leftarrow \text{remainder}
\end{array}$$

$\Rightarrow (x^3 - 3x^2 + 4x - 2) = (x+1)(x^2 - 4x + 8) - 10$.

Also $P(x) = x^3 - 3x^2 + 4x - 2 \Rightarrow P(-1) = -1 - 3 - 4 - 2 = -10$.

(b)

$$\begin{array}{r l}
 & x^2 - (3+i)x + (3i+3) \leftarrow \text{quotient} \\
x+i \,\big) & x^3 - 3x^2 + 4x - 2 \\
 & \underline{x^3 + ix^2} \\
 & (-3-i)\,x^2 + \quad 4x - 2 \\
 & \underline{(-3-i)\,x^2 + (-3i+1)\,x} \\
 & (3i+3)\,x - 2 \\
 & \underline{(3i+3)\,x + 3i - 3} \\
 & 1 - 3i \leftarrow \text{remainder}
\end{array}$$

$\Rightarrow (x^3 - 3x^2 + 4x - 2) = (x+i)\{x^2 - (3+i)x + (3i+3)\} + (1-3i)$.

Also $P(x) = x^3 - 3x^2 + 4x - 2 \Rightarrow P(-i) = i + 3 - 4i - 2 = 1 - 3i$.

5 *Solution*

(a) (i) $P(x) = x^3 + x^2 - 3x - 3 = x^2(x+1) - 3(x+1) = (x+1)(x^2 - 3)$ are irreducible factors over $\mathbf{Q}$.

(ii, iii) Irreducible factors of $P(x)$ over **R** and also **C** are

$P(x)=(x+1)(x-\sqrt{3})(x+\sqrt{3})$.

(b) (i, ii) $P(x)=x^3-2x^2+4x-8=x^2(x-2)+4(x-2)=(x-2)(x^2+4)$ are irreducible factors over **Q** and also **R**.

(iii) Irreducible factors of $P(x)$ over **C** are $P(x)=(x-2)(x-2i)(x+2i)$.

6 *Solution*

(a) (i) The only possible rational zeros of $P(x)$ are $\pm 1, \pm 2, \pm 4, \pm 8$ (integer divisors of the constant term 8). But of these, only 1 and –4 satisfy $P(x)=0$. Hence $(x-1)$ and $(x-4)$ are factors of $P(x)$. By polynomial division $P(x)$ by $(x-1)(x+4)=x^2+3x-4$ we obtain $P(x)=(x-1)(x+4)(x^2-2)$ and these are irreducible factors over **Q**.

(ii, iii) $P(x)=(x-1)(x+4)(x^2-2)=(x-1)(x+4)(x-\sqrt{2})(x+\sqrt{2})$.

(b) (i, ii) The integer divisors of the constant term –6 are $\pm 1, \pm 2, \pm 3, \pm 6$. Of these only –2 and 3 satisfy $P(x)=0$. Polynomial division $P(x)$ by $(x+2)(x-3)$ yields $P(x)=x^4-x^3-5x^2-x-6=(x+2)(x-3)(x^2+1)$, and these are irreducible factors over **Q** and **R**.

(iii) $P(x)=(x+2)(x+3)(x^2+1)=(x+2)(x+3)(x-i)(x+i)$.

7 *Solution*

(a) According to the condition of the problem and factor theorem factors of $P(x)$ are $(x-5)$ and $(x+2)^2$.

Hence $P(x)=(x-5)(x+2)^2=(x-5)(x^2+4x+4)=x^3-x^2-16x-20$.

(b) Factors of $P(x)$ are $(x+1)$ and $(x-3)^3$.

Hence $P(x)=(x+1)(x-3)^3=(x+1)(x^3-9x^2+27x-27)=x^4-8x^3+18x^2-27$.

8 *Solution*

$P(x) = x^3 - 3x^2 + 4 \Rightarrow P'(x) = 3x^2 - 6x \Rightarrow P'(0) = 0$ and $P'(2) = 0$, but

$P(0) \neq 0, P(2) = 0$.

Hence 2 is a multiple zero of $P(x)$. As $P''(2) \neq 0$, its multiplicity is two.

9 *Solution*

Investigate rational roots of $P(x) = 0$. Among integer divisors of the constant term

$a_0 = -2$ of $P(x)$ only -1 and 2 satisfy $P(x) = 0$.

$P'(x) = 4x^3 + 3x^2 - 6x - 5 \Rightarrow P'(2) \neq 0$

and hence 2 is a single zero. $P'(-1) = 0, P''(x) = 12x^2 + 6x - 6 \Rightarrow P''(-1) = 0$.

$P^{(3)}(x) = 24x + 6 \Rightarrow P^{(3)}(-1) \neq 0$. Hence -1 is a root of multiplicity 3 of $P(x) = 0$.

As

$P(x)$ is a polynomial of degree 4, $P(x) = 0$ has no other roots except for 2 and -1.

10 *Solution*

$P(x) = 4x^3 + 12x^2 - 15x + 4$,

$P'(x) = 12x^2 + 24x - 15$,

$P''(x) = 24x + 24$.

$\Rightarrow P'(1/2) = 0, P'(-5/2) = 0$. But $P(1/2) = 0, P(-5/2) \neq 0 \Rightarrow 1/2$ is a multiple

zero. As $P''(1/2) \neq 0, 1/2$ is double zero, and $(2x-1)^2$ is a factor of $P(x)$. By

polynomial division $P(x) = (x+4)(2x-1)^2$. These are irreducible factors over **R**.

-4 is a single zero, $1/2$ is a double zero.

11 *Solution*

$P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$,

$P'(x) = 4x^3 - 9x^2 - 12x + 28$,

$P''(x) = 12x^2 - 18x - 12$,

$P'''(x) = 24x - 18. \Rightarrow P''(2) = 0, P'(2) = 0, P(2) = 0, P^{(3)}(2) \neq 0$.

Hence 2 is a triple zero of $P(x)$ and $P(x)=(x-2)^3(x+k)$ for some constant k, as $P(x)$ is a monic polynomial of degree 4. Then $P(0)=-24 \Rightarrow k=3$ and $P(x)=(x-2)^3(x+3)$. So -3 is a triple zero of $P(x)$.

12 *Solution*

$P(x)=x^3-3x^2-9x+c$,

$P'(x)=3x^2-6x-9$,

$P''(x)=6x-6$.

$\Rightarrow P'(-1)=0, P''(-1)\neq 0, P'(3)=0, P''(3)\neq 0$. Hence both -1 and 3 can be a double zero of $P(x)$.

Let -1 be a double zero of $P(x). \Rightarrow P(x)=(x+1)^2(x+k)$ for some constant k, as $P(x)$ is a monic polynomial of degree 3.

$P(0)=c \Rightarrow k=c. P(-1)=0 \Rightarrow c=-5 \Rightarrow P(x)=(x+1)^2(x-5)$.

Let 3 be a double zero of $P(x). \Rightarrow P(x)=(x-3)^2(x+l)$ for some constant l,

$P(3)=0 \Rightarrow c=27. P(0)=c \Rightarrow l=\frac{c}{9}=3. \Rightarrow P(x)=(x-3)^2(x+3)$.

13 *Solution*

$P(x)=x^4+2x^3-12x^2-40x+c$,

$P'(x)=4x^3+6x^2-24x-40$,

$P''(x)=12x^2+12x-24$,

$P'''(x)=24x+12. \Rightarrow P''(1)=0, P''(-2)=0. P'(1)\neq 0, P'(-2)=0, P^{(3)}(-2)\neq 0$.

Hence -2 is a triple zero of $P(x)$ and $P(x)=(x+2)^3(x+k)$ for some constant k, as $P(x)$ is a monic polynomial of degree 4.

Then $P(-2)=0 \Rightarrow c=-32. P(0)=c \Rightarrow k=\frac{c}{8}=-4$ and $P(x)=(x+2)^3(x-4)$.

14 *Solution*

$P(x)=ax^3+bx^2+d$,

$P'(x)=3ax^2+2bx$,

$P''(x)=6ax+2b$.

$\Rightarrow P'(0)=0,\ P'\left(-\dfrac{2b}{3a}\right)=0$. Hence both 0 and $-2b/(3a)$ can be a double root of $P(x)=0$.

Let 0 be a double root. Hence $P(0)=0 \Rightarrow d=0 \Rightarrow$ if $27a^2d+4b^3=0$, then $b=0 \Rightarrow P(x)=ax^3$ and 3 is a triple root. Thus if 0 is a double root, then $27a^2d+4b^3 \neq 0$.

Let $-2b/(3a)$ be a double root of $P(x)=0$. Hence

$$P(-2b/(3a))=0 \Rightarrow a\left(\frac{-2b}{3a}\right)^3+b\left(\frac{-2b}{3a}\right)^2+d=0 \Rightarrow 27a^2d+4b^3=0.$$

15 *Solution*

$P(x)=ax^3+cx+d$,

$P'(x)=3ax^2+c$,

$P''(x)=6ax$.

$\Rightarrow P'\left(e\sqrt{\dfrac{-c}{3a}}\right)=0$, where $e:=\pm 1$. Hence $e\sqrt{\dfrac{-c}{3a}}$ can be a double root.

$$P\left(e\sqrt{\frac{-3}{3a}}\right)=0 \Rightarrow a\left(e\left(\frac{-c}{3a}\right)^{1/2}\right)^3+ce\left(\frac{-c}{3a}\right)^{1/2}+d=0 \Rightarrow d\left(\frac{-c}{3a}\right)^{-1/2}=\frac{-2}{3}ec \Rightarrow$$

$\Rightarrow 4c^3+27ad^2=0$.

16 *Solution*

If $P(x)=1-x-\dfrac{x^2}{2!}-\text{L}+(-1)^n\dfrac{x^n}{n!}$, then

$$P'(x)=-1+x-\text{L}+(-1)^n\frac{x^{n-1}}{(n-1)!}.\Rightarrow P(x)-P'(x)=2P(x)-(-1)^n\frac{x^n}{n!}.$$

(1)Suppose α is a multiple zero of $P(x)$, then $P(\alpha)=P'(\alpha)=0$, and

$P(\alpha)-P'(\alpha)=0 \Rightarrow (-1)^n \dfrac{\alpha^n}{n!}=0$, using $(1) \Rightarrow \alpha=0$. But $P(0)\neq 0$. Hence $P(x)$ has no multiple zero.

17 *Solution*

(a) The only rational zeros of $P(x)$ are $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$. But of these, only $\frac{3}{2}$ satisfies $P(x)=0$. Hence $(2x-3)$ is a factor of $P(x)$. By polynomial division, $2x^3-3x^2+2x-3=(2x-3)(x^2+1)$, and these are irreducible factors over **R**.

(b) The only rational zeros of $P(x)$ are $\pm 1, \pm 2, \pm\frac{1}{2}$. But of these, only $-\frac{1}{2}$ satisfies $P(x)=0$. Hence $(2x+1)$ is a factor of $P(x)$. By polynomial division, $2x^3+x^2-4x-2=(2x+1)(x^2-2)=(2x+1)(x-\sqrt{2})(x+\sqrt{2})$, and these are irreducible factors over **R**.

18 *Solution*

The only rational zeros of $P(x)$ are $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{3}{4}$. But of these, only $\frac{1}{2}$ and $-\frac{3}{2}$ satisfy $P(x)=0$. Hence $(2x-1)$ and $(2x+3)$ are factors of $P(x)$. By polynomial division, $4x^4+8x^3+5x^2+x-3=(2x-1)(2x+3)(x^2+x+1)$, and these are irreducible factors over **R**, as $(x^2+x+1)=0$ has no roots over **R** and so cannot be factored further over **R**.

Exercise 4.2

1 *Solution*

(a) $x+i$ is a linear divisor. Hence we can use a remainder theorem, and the remainder is

$P(-i)=(-i)^3+2(-i)^2+1=i-2+1=-1+i$.

(b) $P(x)=x^3+2x^2+1$ and $D(x)=x^2+1$ are polynomials over **Q**. By the division transformation, $P(x)\equiv D(x)S(x)+R(x)$ where $R(x)$ is a polynomial over **Q**, such that

deg R< deg D =2. Thus $P(x)\equiv(x^2+1)S(x)+ax+b,\quad a,b$ rational.

$\Rightarrow P(i)=0+ai+b$, and hence $ai+b=i^3+2i^2+1=-1-i\Rightarrow a=-1, b=-1$. Hence the remainder $ax+b$ is $-x-1$.

2 *Solution*

(a) $x-2i$ is a linear divisor. Hence we can use a remainder theorem, and the remainder is

$P(2i)=(2i)^5-3(2i)^4+2(2i)-1=32i-48+4i-1=-49+36i$.

(b) $P(x)=x^5-3x^4+2x-1$ and $D(x)=x^2+4$ are polynomials over **Q**.

Hence $P(x)=D(x)S(x)+R(x)$, where $R(x)$ is rational and deg R< deg D =2. Thus

$P(x)\equiv(x^2+4)S(x)+ax+b,\quad a,b$ rational. $\Rightarrow P(2i)=0+2ai+b$, and hence

$2ai+b=(2i)^5-3(2i)^4+2(2i)-1=32i-48+4i-1=36i-49\Rightarrow a=18, b=-49$.

Hence the remainder $ax+b$ is $18x-49$.

3 *Solution*

By the division transformation, $x^4+ax^2+2x=(x^2+1)S(x)+2x+3$. Substituting

$x=i,\quad 1-a+2i=2i+3\Rightarrow a=-2$.

4 *Solution*

By the division transformation, $x^4+ax^3+3x-11=(x^2+4)\,Q\,(x)+x+5$.

Substituting $x=2i$, $16-8ai+6i-11=2i+5 \Rightarrow a=1/2$.

5 *Solution*

By the division transformation, $x^4+ax^2+bx+2=(x^2+1)S(x)-x+1$. Substituting $x=i$, $1-a+bi+2=-i+1$, that is $-a+bi=-i-2 \Rightarrow a=2, b=-1$.

6 *Solution*

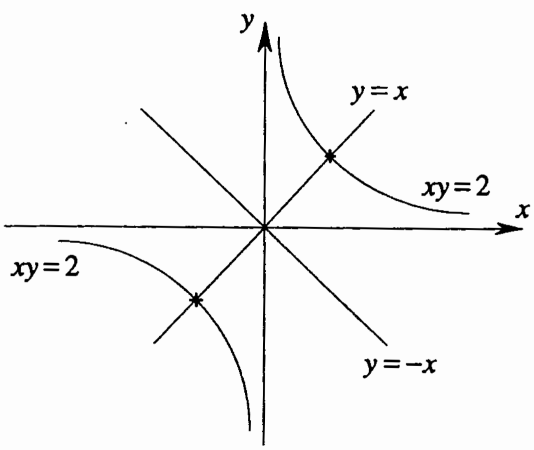
By the division transformation, $x^4+ax^3+b=(x^2+4)S(x)-x+13$. Substituting $x=2i$, we obtain $16-8ai+b=-2i+13$, that is $-8ai+b=-2i-3 \Rightarrow a=1/4, b=-3$.

7 *Solution*

If $z=x+iy$, then

$z^2=y^2-x^2+2xyi=4i \Rightarrow y^2-x^2=0$,

$xy=2$. $x^2-y^2=(x-y)(x+y) \Rightarrow x^2-y^2=0$ if and only if $y=x$ or $y=-x$. Hence as seen from the graphs there are two points of intersection.



8 *Solution*

Consider the polynomial $P_1(z)-P_2(z)$ which has more than n zeros. But deg $(P_1-P_2)\le n$. It is possible only if $P_1(z)-P_2(z)=0$ identically. Hence

$P_1(0)-P_2(0)=0 \Rightarrow b_0=c_0$, $P_1'(0)-P_2'(0)=0 \Rightarrow b_1=c_1$,K $,P_1^{(n)}(0)-P_2^{(n)}(0)=0 \Rightarrow b_n=c$

9 *Solution*

$P(x)$ has real coefficients. Hence $P(i)=0 \Rightarrow P(-i)=0$ and then

$(x-i)(x+i)=x^2+1$ is a factor of $P(x)$. By the division transformation

$P(x)=x^4+x^3-x^2+x-2=(x^2+1)(x^2+x-2). \Rightarrow P(x)=(x^2+1)(x-1)(x+2)$. This is the factorisation of $P(x)$ into irreducible factors over **R**, and $P(x)$ has zeros $i, -i, -2$ and 1 over **C**.

10 *Solution*

$P(x)$ has real coefficients. Hence $P(2-i)=0 \Rightarrow P(2+i)=0$ and then $[x-(2-i)]\cdot[x-(2+i)]=x^2-4x+5$ is a factor of $P(x)$. By the division transformation

$P(x)=x^4-5x^3+7x^2+3x-10=(x^2-4x+5)(x^2-x-2). \Rightarrow P(x)=(x^2-4x+5)(x+1)(x-2)$. This is the factorisation of $P(x)$ into irreducible factors over **R**, and $P(x)$ has zeros $2-i, 2+i, -1$ and 2 over **C**.

11 *Solution*

$P(x)$ has real coefficients. Hence $P(1+2i)=0 \Rightarrow P(1-2i)=0$ and then $[x-(1+2i)]\cdot[x-(1-2i)]=x^2-2x+5$ is a factor of $P(x)$. By the division transformation $P(x)=x^4-2x^3+6x^2-2x+5=(x^2-2x+5)(x^2+1)$.

This is the factorisation of $P(x)$ into irreducible factors over **R**, and $P(x)$ has zeros $(1+2i),(1-2i),-i$ and i over **C**.

12 *Solution*

$P(x)$ has real coefficients. Hence $P(i)=0 \Rightarrow P(-i)=0$, and then $(x-i)(x+i)=x^2+1$ is a factor of $P(x)$. The rational zero of $P(x)$ is p/q, where q is a divisor of the leading coefficient 1 and p is a divisor of the constant term 3.

Hence $P(x)$ has the form $(x^2+1)(x-\alpha)(x-\beta)$, where the rational zero α takes one of the values $\pm 1, \pm 2,$ or ± 3 (since $P(x)$ is a monic polynomial of degree 4). Given that the constant term is $3 \Rightarrow \alpha\,\beta = 3$, and hence the zeros of $P(x)$ are $i, -i, 1$ and 3 or $i, -i, -1$ and -3. But the sum of the zeros is negative. Thus $P(x)=(x^2+1)(x+1)(x+3)$, and these factors are irreducible over **R**.

13 *Solution*

$P(x)$ is an even monic polynomial of degree 4. Hence $P(x)=x^4+ax^2+b$. $P(x)$ has real coefficients. Hence $P(2i)=0 \Rightarrow P(-2i)=0$ and then $(x-2i)(x+2i)=x^2+4$ is a factor of

$P(x) \Rightarrow P(x)=(x^2+4)(x^2+c)$.The product of zeros of $P(x)$ is -8. Hence $4c=-8 \Rightarrow c=-2$, and $P(x)=(x^2+4)(x^2-2)=(x^2+4)(x-\sqrt{2})(x+\sqrt{2})$. These are irreducible factors of $P(x)$ over **R**, and $P(x)$ has zeros $-2i, +2i, -\sqrt{2}$ and $\sqrt{2}$ over **C**.

14 *Solution*

$P(x)=x^4+ax^3+bx^2+cx+4$.

$P(\sqrt{2})=0 \Rightarrow (8+2b)+(2a+c)\sqrt{2}=0,\quad a,\ b,\ c$ interger $\Rightarrow 8+2b=0$ and $2a+c=0$.

Hence

$P(x)=x^4+ax^3-4x^2-2ax+4=(x^2-2)^2+ax(x^2-2)=(x^2-2)(x^2+ax-2)$.

Thus $x^2+ax-2=(x-\alpha)(x+\beta)$ has a rational zero, which may be $\pm 1, \pm 2$.

But $\alpha\beta=-2 \Rightarrow \alpha=-1, \beta=2$ or $\alpha=1, \beta=-2$. The sum of zeros of $P(x)$ is positive.

Hence $-\sqrt{2}+\sqrt{2}+\alpha+\beta=\alpha+\beta>0 \Rightarrow \alpha=-1, \beta=2$.

Thus $P(x)=(x^2-2)(x+1)(x-2)=(x-\sqrt{2})(x+\sqrt{2})(x+1)(x-2)$, and this is the factorisation of $P(x)$ into irreducible factors over **R**.

Exercise 4.3

1 *Solution*

$P(x)=x^3+ax^2+bx+c$, since $P(x)$ is the monic of degree three. If $\alpha=1, \beta=2, \gamma=3$ denote the zeros of $P(x)$, then

$a=-\sum\alpha=-(1+2+3)=-6,$

$b=\sum\alpha\beta=2+3+6=11,$

$c=-\sum\alpha\beta\gamma=-6.$

Hence $P(x)=x^3-6x^2+11x-6.$

2 *Solution*

$P(x)=x^4+ax^3+bx^2+cx+d$, since $P(x)$ is monic of degree four. If $\alpha=-3, \beta=-1, \gamma=1$ and $\delta=3$ denote the zeros of $P(x)$, then

$a=-\sum\alpha=-(-3-1+1+3)=0,$

$b=\sum\alpha\beta=3-3-9-1-3+3=-10,$

$c=-\sum\alpha\beta\gamma=-(3+9-9-3)=0,$

$d=\sum\alpha\beta\gamma\delta=9.$

Hence $P(x)=x^4-10x^2+9.$

3 *Solution*

Let the roots of $P(x)$ be $\alpha, \frac{1}{\alpha}, \beta$. Then product of roots is $\beta. \Rightarrow \beta=-\frac{-6}{3}=2$. Sum of products taken two at a time is $1+\frac{2}{\alpha}+2\alpha=\frac{23}{3}\Rightarrow\alpha=3,\frac{1}{3}$.

Sum of the roots is $3+\frac{1}{3}+2=\frac{-a}{3}\Rightarrow a=-16$. Hence the roots are $3, \frac{1}{3}, 2$ and the coefficient a is -16.

4 *Solution*

Let the roots of $P(x)=x^3-3x^2-4x+a$ be $\alpha, -\alpha, \beta$.

Then sum of the roots is $\beta. \Rightarrow \beta=3$. Sum of products taken two at a time is

$-\alpha^2+3\alpha-3\alpha=-\alpha^2\Rightarrow-\alpha^2=-4\Rightarrow\alpha=-2,2$.

Product of the roots is $-2\cdot 2\cdot 3=-a \Rightarrow a=12$.

Hence the roots are $-2, 2, 3$ and the coefficient $a=12$.

5 *Solution*

Let the roots of $P(x)=x^4+px^3+qx^2+rx+s$ be $\alpha, \frac{1}{\alpha}, \beta, -\beta$.

Then

$$s=\sum \alpha\beta\gamma\delta=\alpha\cdot\frac{1}{\alpha}\cdot\beta\cdot(-\beta)=-\beta^2 \Rightarrow \beta^2=-s.$$

Hence

$$q=\sum \alpha\beta=\alpha\cdot\frac{1}{\alpha}+\alpha\cdot\beta-\alpha\cdot\beta+\frac{1}{\alpha}\cdot\beta-\frac{1}{\alpha}\cdot\beta-\beta^2=1-\beta^2 \Rightarrow q=1-\beta^2=1+s.$$

$$-p=\sum \alpha=\alpha+\frac{1}{\alpha}+\beta-\beta \Rightarrow \alpha+\frac{1}{\alpha}=-p.$$

Hence

$$-r=\sum \alpha\beta\gamma=\alpha\cdot\frac{1}{\alpha}\cdot\beta+\alpha\cdot\frac{1}{\alpha}\cdot(-\beta)+\alpha\cdot\beta\cdot(-\beta)+\frac{1}{\alpha}\cdot\beta\cdot(-\beta)=-\left(\alpha+\frac{1}{\alpha}\right)\beta^2 \Rightarrow$$

$$r=-p(-s)=ps.$$

6 *Solution*

The sum of roots is $\frac{-q}{p}=\sum \alpha=(a-c)+a+(a+c)=3a \Rightarrow a=\frac{-q}{3p}$.

Hence $0=P(a)=P\left(\frac{-q}{3p}\right)=\frac{-p\cdot q^3}{27p^3}+\frac{q\cdot q^2}{9p^2}-\frac{rq}{3p}-s \Rightarrow 0=P(a)\cdot 27p^2$

$=2q^3-9pqr+27p^2s$.

7 *Solution*

Let the roots be $a-c, a, a+c$. Then

$$-\frac{27}{18}=\sum \alpha=(a-c)+a+(a+c)=3a \Rightarrow a=-\frac{1}{2}.$$

$\frac{4}{18}=\sum \alpha\cdot\beta\cdot\gamma=(a-c)\,a\,(a+c)=-\frac{1}{2}\left(\frac{1}{4}-c^2\right) \Rightarrow c=\frac{5}{6}$ (or $c=-\frac{5}{6}$ that gives the same roots). Hence the roots are $a-c=-\frac{4}{3}$, $a=-\frac{1}{2}$ and $a+c=\frac{1}{3}$.

8 *Solution*

Let the roots be $b-c, b, b+c$. Then $6=\sum\alpha=(b-c)+b+(b+c)=3b \Rightarrow b=2$.

$-10=\sum\alpha\cdot\beta\cdot\gamma=(b-c)\,b\,(b+c)=2(4-c^2)\Rightarrow c=3$ (or $c=-3$ that gives the same values of the roots and so the same constant a).

Hence the roots are $b-c=-1, b=2, b+c=5$ and $a=\sum\alpha\cdot\beta=-2-5+10=3$.

9 *Solution*

The product of the roots is $-\frac{s}{p}=\sum\alpha\cdot\beta\cdot\gamma=ac\cdot a\cdot\frac{a}{c}=a^3\Rightarrow a=\sqrt[3]{(-s/p)}$.

The sum of the roots is $\frac{-q}{p}=\sum\alpha=a\cdot c+a+\frac{a}{c}=a\left(c+1+\frac{1}{c}\right)$, and the product of the roots taken two at a time is $\frac{r}{p}=\sum\alpha\cdot\beta=a^2c+a^2+\frac{a^2}{c}=a^2\left(c+1+\frac{1}{c}\right)$.

Hence $\frac{-q}{p}a=\frac{r}{p}\Rightarrow -q\cdot\sqrt[3]{(-s/p)}=r\Rightarrow pr^3-q^3s=0$.

10 *Solution*

Let the roots be $a\cdot c, a, \frac{a}{c}$. Then

$$-\frac{16}{2}=\sum\alpha\cdot\beta\cdot\gamma=ac\cdot a\cdot\frac{a}{c}=a^3\Rightarrow a=-2,$$
$$\frac{13}{2}=\sum\alpha=a\cdot c+a+\frac{a}{c}=-2\left(c+1+\frac{1}{c}\right)\Rightarrow$$
$$c+\frac{1}{c}=\frac{-17}{4}\Rightarrow 4c^2+17c+4=0\Rightarrow c=-\frac{1}{4}$$

(or $c=-4$ that gives the same roots). Hence the roots are $a\cdot c=\frac{1}{2}, a=-2, \frac{a}{c}=8$.

11 *Solution*

Let the roots be $b\cdot c, b, \frac{b}{c}$. Then

$$27=\sum\alpha\cdot\beta\cdot\gamma=bc\cdot b\cdot\frac{b}{c}=b^3\Rightarrow b=3,$$

$$13=\sum\alpha=b\cdot c+b+\frac{b}{c}=3\left(\frac{1}{c}+1+c\right)\Rightarrow$$

$$\frac{1}{c}+c=\frac{10}{3}\Rightarrow 3c^2-10c+3=0,$$

and using quadratic formula $c=3$ (or $c=1/3$ that gives rise to the same values of the roots and so of the constant a).

Hence the roots are $b\cdot c=9, b=3, \frac{b}{c}=1$ and $a=\sum\alpha\cdot\beta=27+9+3=39$.

12 *Solution*

(a) The values $2\alpha, 2\beta$ and 2γ satisfy $\left(\frac{x}{2}\right)^3+3\left(\frac{x}{2}\right)^2-2\left(\frac{x}{2}\right)-2=0$ and hence the required equation is $x^3+6x^2-8x-16=0$.

(b) The values $\alpha-2, \beta-2$ and $\gamma-2$ satisfy $(x+2)^3+3(x+2)^3-2(x+2)-2=0$ and hence the required equation is $x^3+9x^2+22x+14=0$.

(c) The values $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ satisfy $\left(\frac{1}{x}\right)^3+3\left(\frac{1}{x}\right)^2-2\left(\frac{1}{x}\right)-2=0$ and hence the required equation is $2x^3+2x^2-3x-1=0$.

(d) The values α^2, β^2 and γ^2 satisfy $\left(x^{1/2}\right)^3+3\left(x^{1/2}\right)^2-2x^{1/2}-2=0$.

Rearrangement gives

$x^{1/2}(x-2)=2-3x$. Squaring we obtain $x(x-2)^2=(2-3x)^2$ and hence the required equation is $x^3+9x^2+22x+14=0$.

13 *Solution*

(a) The values $2\alpha, 2\beta, 2\gamma$ and 2δ satisfy the equation

$\left(\frac{x}{2}\right)^4+4\left(\frac{x}{2}\right)^3-3\left(\frac{x}{2}\right)^2-4\left(\frac{x}{2}\right)+2=0$. Hence the required equation is

$x^4+8x^3-12x^2-32x+32=0$.

(b) $\alpha-2, \beta-2, \gamma-2$ and $\delta-2$ satisfy

$(x+2)^4+4(x+2)^3-3(x+2)^2-4(x+2)+2=0$. Hence the required equation is

$x^4+12x^3+45x^2+64x+30=0$.

(c) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ satisfy $\left(\frac{1}{x}\right)^4+4\left(\frac{1}{x}\right)^3-3\left(\frac{1}{x}\right)^2-4\left(\frac{1}{x}\right)+2=0$. Hence the

required equation is $2x^4-4x^3-3x^2+4x+1=0$.

(d) $\alpha^2, \beta^2, \gamma^2$ and δ^2 satisfy $\left(x^{1/2}\right)^4+4\left(x^{1/2}\right)^3-3\left(x^{1/2}\right)^2-4x^{1/2}+2=0$.

Rearrangement gives $x^{1/2}(4x-4)=-x^2+3x-2$. Squaring and simplifying, the

required equation is $x^4-22x^3+45x^2-28x+4=0$.

14 *Solution*

(a) $\alpha^2 \cdot \beta \cdot \gamma, \alpha \cdot \beta^2 \cdot \gamma$ and $\alpha \cdot \beta \cdot \gamma^2$ can be rewritten $\alpha\beta\gamma \cdot \alpha, \alpha\beta\gamma \cdot \beta$ and $\alpha\beta\gamma \cdot \gamma$. But

$\alpha\beta\gamma=3$.

Hence the required equation has the roots $3\alpha, 3\beta$ and 3γ, which satisfy

$\left(\frac{x}{3}\right)^3+\left(\frac{x}{3}\right)^2-2\left(\frac{x}{3}\right)-3=0$. And the required equation is $x^3+3x^2-18x-81=0$.

(b) $\alpha+\beta+\gamma=-1$. Hence the required equation has the roots $\alpha-1, \beta-1$ and $\gamma-1$

which

satisfy $(x+1)^3+(x+1)^2-2(x+1)-3=0$. The required equation is

$x^3+4x^2+3x-3=0$.

15 *Solution*

(a) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ satisfy $\left(\frac{1}{x}\right)^3+2\left(\frac{1}{x}\right)+1=0$. Hence the required equation is

$x^3+2x^2+1=0$.

(b) α^2, β^2 and γ^2 satisfy $\left(x^{1/2}\right)^3+2x^{1/2}+1=0$. Rearrangement gives

$x^{1/2}(x+2)=-1$. Squaring and simplifying, the required equation is

$x^3+4x^2+4x-1=0$.

(c) From (b) α^2, β^2 and γ^2 satisfy $x^3+4x^2+4x-1=0$, and hence

$\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$

satisfy $\left(\dfrac{1}{x}\right)^3+4\left(\dfrac{1}{x}\right)^2+4\left(\dfrac{1}{x}\right)-1=0$. And the required equation is

$x^3-4x^2-4x-1=0$.

16 *Solution*

(a) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$ satisfy $\left(\dfrac{1}{x}\right)^3+p\left(\dfrac{1}{x}\right)^2+r=0$. Hence the required equation is

$rx^3+px^2+1=0$.

(b) α^2, β^2 and γ^2 satisfy $\left(x^{1/2}\right)^3+p\left(x^{1/2}\right)^2+r=0$. Rearrangement gives

$x^{1/2}x=-r-px$.

Squaring and simplifying, the required equation is $x^3-p^2x^2-2prx-r^2=0$.

(c) From (b) α^2, β^2 and γ^2 satisfy $x^3-p^2x^2-2prx-r^2=0$. Hence

$\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$

satisfy $\left(\dfrac{1}{x}\right)^3-p^2\left(\dfrac{1}{x}\right)^2-2pr\left(\dfrac{1}{x}\right)-r^2=0$. Simplifying, the required equation is

$r^2x^3+2prx^2+p^2x-1=0$.

17 *Solution*

(a) $\alpha+\beta+\gamma=\sum\alpha=-1$.

(b) α^2, β^2, and γ^2 are roots of the equation $\left(x^{1/2}\right)^3+\left(x^{1/2}\right)^2+2=0$.

Rearrangement gives $x^{1/2}x=-2-x$. Squaring and simplifying, $x^3-x^2-4x-4=0$.

Hence $\alpha^2+\beta^2+\gamma^2=1$.

(c) $\alpha^3+\alpha^2+2=0$, (since α, β, γ are roots of the given equation)

$\beta^3+\beta^2+2=0$,

$\gamma^3 + \gamma^2 + 2 = 0$.

Hence $(\alpha^3 + \beta^3 + \gamma^2) + (\alpha^2 + \beta^2 + \gamma^2) + 6 = 0$.

From (b) $\alpha^2 + \beta^2 + \gamma^2 = 1$, therefore $\alpha^3 + \beta^3 + \gamma^3 = -7$.

(d) From (b) α^2, β^2, and γ^2 satisfy $x^3 - x^2 - 4x - 4 = 0$.

Hence $\alpha^4 = \left(\alpha^2\right)^2$, $\beta^4 = \left(\beta^2\right)^2$ and $\gamma^4 = \left(\gamma^2\right)^2$ satisfy $\left(x^{1/2}\right)^3 - \left(x^{1/2}\right)^2 - 4x^{1/2} - 4 = 0$.

Rearrangement gives $x^{1/2}(x-4) = x+4$. Squaring and simplifying,

$x^3 - 9x^2 + 8x - 16 = 0$.

$\alpha^4 + \beta^4 + \gamma^4$ is the sum of roots of this equation. Hence $\alpha^4 + \beta^4 + \gamma^2 = 9$.

18 *Solution*

(a) α^2, β^2, and γ^2 satisfy $\left(x^{1/2}\right)^3 + qx^{1/2} + r = 0$.

Rearrangement gives $x^{1/2}(x+q) = -r$. Squaring and simplifying,

$x^3 + 2qx^2 + q^2x - r^2 = 0$.

$\alpha^2 + \beta^2 + \gamma^2$ is the sum of roots of this equation. Hence $\alpha^2 + \beta^2 + \gamma^2 = -2q$.

(b) $\alpha^3 + q\alpha + r = 0$, (since α, β, and γ are the roots of the given equation)

$\beta^3 + q\beta + r = 0$,

$\gamma^3 + q\gamma + r = 0$.

Hence $(\alpha^3 + \beta^3 + \gamma^3) + q(\alpha + \beta + \gamma) + 3r = 0$. Here $\alpha + \beta + \gamma$ is the sum of the roots of the equation $x^3 + qx + r = 0 \Rightarrow \alpha + \beta + \gamma = 0$. Therefore $\alpha^3 + \beta^3 + \gamma^3 = -3r$.

(c) α, β, γ are also roots of the equation $x^2(x^3 + qx + r) = 0$, i.e. $x^5 + qx^3 + rx^2 = 0$.

Hence $\alpha^5 + q\alpha^3 + r\alpha^2 = 0$, $\beta^5 + q\beta^3 + r\beta^2 = 0$ and $\gamma^5 + q\gamma^3 + r\gamma^2 = 0$. Adding these equalities we obtain $(\alpha^5 + \beta^5 + \gamma^5) + q(\alpha^3 + \beta^3 + \gamma^3) + r(\alpha^2 + \beta^2 + \gamma^2) = 0$.

But from (a) $\alpha^2 + \beta^2 + \gamma^2 = -2q$ and from (b) $\alpha^3 + \beta^3 + \gamma^3 = -3r$.

Hence $\alpha^5 + \beta^5 + \gamma^5 = -q(-3r) - r(-2q) = 5qr$.

Exercise 4.4

1 *Solution*

$P(-i) = i - i + 4i - 4i = 0 \Rightarrow (x+i)$ is a factor of $P(x)$. By inspection, or by polynomial division, $x^3 + ix^2 - 4x - 4i = (x+i)(x^2 - 4)$. Hence $P(x) = (x+i)(x-2)(x+2)$, and these are irreducible factors over **C**.

2 *Solution*

$P(2i) = -8i + 8i - 6i + 6i = 0 \Rightarrow (x - 2i)$ is a factor of $P(x)$. By inspection, or by polynomial

division, $x^3 - 2ix^2 - 3x + 6i = (x-2i)(x^2 - 3)$.

Hence $P(x) = (x-2i)(x-\sqrt{3})(x+\sqrt{3})$, and these factors are irreducible over **C**.

3 *Solution*

$$P(x) = x^2\left(3x^2 + 10x + 6 + \frac{10}{x} + \frac{3}{x^2}\right) = x^2\left\{3\left(x^2 + \frac{1}{x^2}\right) + 10\left(x + \frac{1}{x}\right) + 6\right\}.$$

Using $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2$, $P(x) = x^2\left\{3\left(x + \frac{1}{x}\right)^2 + 10\left(x + \frac{1}{x}\right)^2\right\}$. Since 0 is not a zero of $P(x)$, the solutions of $P(x) = 0$ are the solutions of

$$3\left(x + \frac{1}{x}\right)^2 + 10\left(x + \frac{1}{x}\right) = 0.$$

By factorising this quadratic

$$P(x) = x^2\left(x + \frac{1}{x}\right)\left\{3\left(x + \frac{1}{x}\right) + 10\right\} = (x^2+1)(3x^2+10x+3). \qquad (1)$$

Hence

$$P(x) = 0 \Rightarrow \quad x^2 + 1 = 0 \text{ or } 3x^2 + 10x + 3 = 0$$

$$x = \pm i \qquad x = \frac{-5 \pm 4}{3}$$

$$x = -3 \text{ or } x = -\frac{1}{3}.$$

Therefore, the zeros of $P(x)$ are $-3, -\frac{1}{3}, \pm i$.

From (1) $P(x) = (x^2+1)\,3\,(x+3)(x+1/3) = (x^2+1)(x+3)(3x+1)$ over $\mathbf{R}$.

4 *Solution*

$$P(x) = x^2\left(2x^2+7x+2-\frac{7}{x}+\frac{2}{x^2}\right) = x^2\left\{2\left(x^2+\frac{1}{x^2}\right)+7\left(x-\frac{1}{x}\right)+2\right\}.$$

Using $\left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2$, $P(x) = x^2\left\{2\left(x-\frac{1}{x}\right)^2+7\left(x-\frac{1}{x}\right)+6\right\}$. Since 0 is a zero of $P(x)$, the solutions of $P(x)=0$ are the solutions of

$$2\left(x-\frac{1}{x}\right)^2+7\left(x-\frac{1}{x}\right)+6=0.$$

By factorising this quadratic

$$P(x) = x^2\cdot 2\left\{\left(x-\frac{1}{x}\right)+2\right\}\left\{\left(x-\frac{1}{x}\right)+\frac{3}{2}\right\} = (x^2+2x-1)(2x^2+3x-2). \qquad (1)$$

Hence

$$P(x)=0 \Rightarrow \quad x^2+2x-1=0 \text{ or } 2x^2+3x-2=0.$$

$$x=-1\pm\sqrt{2} \qquad x=\frac{-3\pm 5}{4}$$

$$x=-2 \text{ or } x=\frac{1}{2}.$$

Therefore, the roots of $P(x)=0$ are $-2, \frac{1}{2}, -1\pm\sqrt{2}$.

From (1) $P(x) = (x+1-\sqrt{2})(x+1+\sqrt{2})(x+2)(2x-1)$ over $\mathbf{R}$.

5 *Solution*

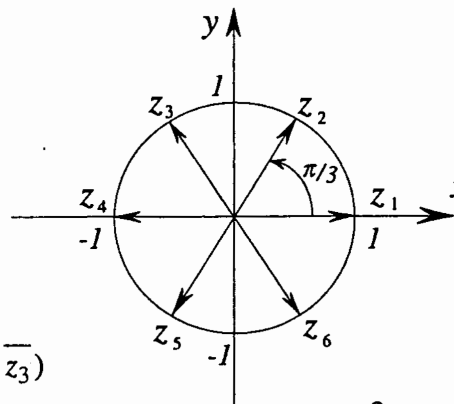
$$x^6-1 = (x-1)(x^5+x^4+x^3+x^2+x+1) = (x-1)\left\{x(x^4+x^2+1)+(x^4+x^2+1)\right\}$$

$$=(x-1)(x+1)(x^4+x^2+1).$$

Hence $x^6-1=0 \Rightarrow x=\pm 1$ or $x^4+x^2+1=0$. Further more, the sixth roots of unity are equally spaced by $\frac{\pi}{3}$ around a circle of radius 1 and centre $(0,0)$ in the Argand diagram. The zeros of $P(x)=x^4+x^2+1$ are the non-real sixth roots of unity which are:

z_2 and $z_6=\overline{z_2}$, where $z_2=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$,

z_3 and $z_5=\overline{z_3}$, where $z_3=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$.



Hence

$$P(x)=x^4+x^2+1=(x-z_2)(x-\overline{z_2})(x-z_3)(x-\overline{z_3})$$

$$=(x^2-2\operatorname{Re}z_2x+|z_2|^2)(x^2-2\operatorname{Re}z_3x+|z_3|^2)=(x^2-2\cos\frac{\pi}{3}x+1)(x^2-2\cos\frac{2\pi}{3}x+1).$$

Using $\cos\frac{\pi}{3}=\frac{1}{2}$, $\cos\frac{2\pi}{3}=-\cos\frac{\pi}{3}=-\frac{1}{2}$, $x^4+x^2+1=(x^2-x+1)(x^2+x+1)$.

These factors are irreducible over **R**.

6 *Solution*

Let $Q(x)=x^6+1$, then $Q(\pm i)=0$ and hence $(x-i)(x+i)=x^2+1$ is a factor of $Q(x)$.

By inspection, or by polynomial division, $x^6+1=(x^2+1)(x^4-x^2+1)$. Hence

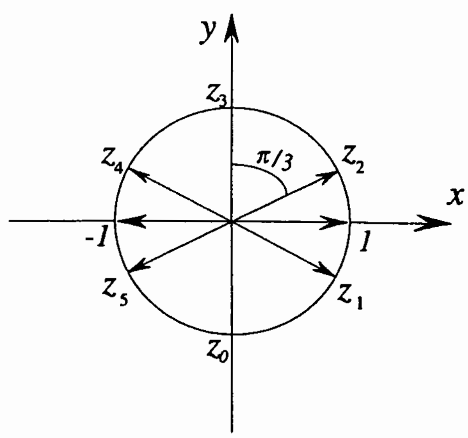
$x^6+1=0\Rightarrow$

$x=\pm i$ or $x^4-x^2+1=0$. Therefore the zeros of $P(x)=x^4-x^2+1$ are the solutions of $x^6=-1, x\neq\pm i$.

The sixth roots of -1 are equally spaced by $\frac{\pi}{3}$ around a circle of radius 1 and centre (0,0) in the Argand diagram. The sixth roots of -1 different from $\pm i$ are:

$z_2, z_1=\overline{z_2}$, where $z_2=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$,

$z_4, z_5=\overline{z_4}$, where $z_4=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}$.



Hence

$$P(x) = x^4 - x^2 + 1 = (x - z_2)(x - \overline{z_2}) \cdot (x - z_4)(x - \overline{z_4})$$
$$(x^2 - 2\,\mathrm{Re}\, z_2 x + |z_2|^2)(x^2 - 2\,\mathrm{Re}\, z_4 x + |z_4|^2) = (x^2 - 2\cos\frac{\pi}{6}x + 1)(x^2 - 2\cos\frac{5\pi}{6}x + 1).$$

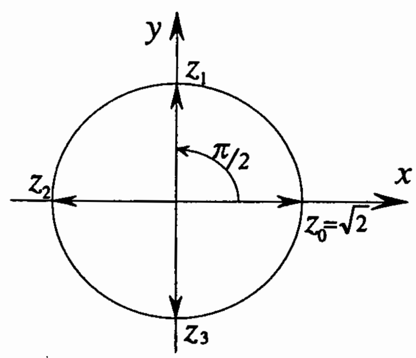
Using

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \cos\frac{5\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}, \quad x^4 - x^2 + 1 = (x^2 - \sqrt{3}\,x + 1)(x^2 + \sqrt{3}\,x + 1).$$

These factors are irreducible over **R**.

7 *Solution*

$z^5 - 4z = 0 \Rightarrow z(z^4 - 4) = 0$. Hence $z = 0$ or z is a complex fourth root of 4. Clearly two such roots are $\pm\sqrt[4]{4} = \pm\sqrt{2}$. The other fourth roots of 4 are equally spaced by $\frac{\pi}{2}$ around a circle of radius $\sqrt{2}$ and centre $(0,0)$ in the Argand diagram.



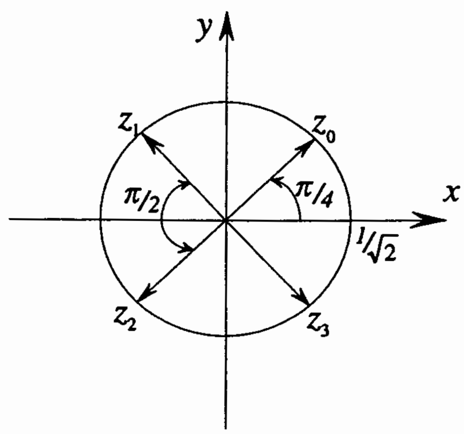
The fourth roots of 4 are

$z_0 = \sqrt{2},\ z_2 = -\sqrt{2},\ z_1 = \sqrt{2}i,\ z_3 = -\sqrt{2}i$.

Hence $z^5 - 4z = 0$ has roots $0, \pm\sqrt{2}, \pm\sqrt{2}i$.

8 *Solution*

$4z^5 + z = 0 \Rightarrow\ z\left(z^4 + \frac{1}{4}\right) = 0$. Hence $z = 0$ or z is a complex fourth root of $-\frac{1}{4}$. Clearly one such root has argument $\frac{\pi}{4}$ and modulus $\frac{1}{\sqrt{2}}$, since $\arg\left(-\frac{1}{4}\right) = \pi$ and $\left|-\frac{1}{4}\right| = \left(\frac{1}{\sqrt{2}}\right)^4$. The other fourth roots of $-\frac{1}{4}$ are equally spaced by $\frac{\pi}{2}$ around a circle of radius $\frac{1}{\sqrt{2}}$ and centre $(0,0)$ in the Argand diagram.



The fourth roots of $-\frac{1}{4}$ are $\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}\right)$ and $\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} \pm i\sin\frac{3\pi}{4}\right)$.

Hence $4z^5+z=0$ has roots $0, \frac{1}{2}(1\pm i), \frac{1}{2}(-1\pm i)$.

9 *Solution*

Let $z=\cos\theta+i\sin\theta$. Then by *De Moivre's* theorem, $z^4=\cos 4\theta+i\sin 4\theta$. But by the Binomial theorem, $z^4=(\cos\theta+i\sin\theta)^4=\sum_{k=0}^{4}\binom{4}{k}i^k\sin^k\theta\cos^{4-k}\theta$. Equating real parts,

$\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta=8\cos^4\theta-8\cos^2\theta+1$.

(a) Let $\cos\theta=x$. Then $\cos 4\theta=0\Leftrightarrow 8x^4-8x^2+1=0$. Hence if θ is a solution of $\cos 4\theta$,

$\cos\theta$ is a root of $8x^4-8x^2+1=0$.

But $\cos 4\theta=0\Rightarrow 4\theta=\pm\frac{\pi}{2}+2\pi n$, n integral $\theta=\pm\frac{\pi}{8}+\frac{\pi}{2}n, n=0,\pm1,\pm2,K$

These values of θ give exactly four distinct values of $\cos\theta$, namely

$\cos\frac{\pi}{8},\cos\frac{5}{8}\pi=-\cos\frac{3}{8}\pi,\cos\frac{9}{8}\pi=-\cos\frac{\pi}{8},\cos\frac{13}{8}\pi=\cos\frac{3}{8}\pi$.

At the same time considering $8x^4-8x^2+1=0$ as a quadratic in x^2,

$x^2=\frac{4\pm\sqrt{8}}{8}=\frac{2\pm\sqrt{2}}{4}$,

$x=\pm\frac{1}{2}\sqrt{2+\sqrt{2}}$ or $x=\pm\frac{1}{2}\sqrt{2-\sqrt{2}}$.

But $x=\cos\theta$. Since $\cos\frac{\pi}{8}>\cos\frac{3\pi}{8}>0$, we deduce $\cos\frac{\pi}{8}=\frac{1}{2}\sqrt{2+\sqrt{2}}$. Since $0>\cos\frac{5\pi}{8}>\cos\frac{9\pi}{8}$, we deduce $\cos\frac{5\pi}{8}=-\frac{1}{2}\sqrt{2-\sqrt{2}}$.

(b) Let $\cos\theta=x$. Then $\cos 4\theta=\frac{1}{2}\Leftrightarrow 8x^4-8x^2+1=\frac{1}{2}$. Hence if θ is a solution of $\cos 4\theta=\frac{1}{2}$, $\cos\theta$ is a root of $16x^4-16x^2+1=0$.

But $\cos 4\theta=\frac{1}{2}\Rightarrow 4\theta=\pm\frac{\pi}{3}+2\pi n$, n integral $\theta=\pm\frac{\pi}{12}+\frac{\pi}{2}n, n=0,\pm1,\pm2,K$

These values of θ give exactly four distinct values of $\cos\theta$, namely

$$\cos\frac{\pi}{12},\cos\frac{5}{12}\pi,\cos\frac{7}{12}\pi=-\cos\frac{5}{12}\pi,\cos\frac{13}{12}\pi=-\cos\frac{\pi}{12}.$$

At the same time considering $16x^4-16x^2+1=0$ as a quadratic in x^2,

$$x^2=\frac{8\pm\sqrt{48}}{16}=\frac{2\pm\sqrt{3}}{4},$$

$$x=\pm\frac{1}{2}\sqrt{2+\sqrt{3}}\text{ or }x=\pm\frac{1}{2}\sqrt{2-\sqrt{3}}.$$

But $x=\cos\theta$. Since $\cos\frac{\pi}{12}>\cos\frac{5\pi}{12}>0$, we deduce that

$$\cos\frac{\pi}{12}=\frac{1}{2}\sqrt{2+\sqrt{3}}\,,\cos\frac{5\pi}{12}=\frac{1}{2}\sqrt{2-\sqrt{3}}\,.$$

10 *Solution*

Let $z=\cos\theta+i\sin\theta$. Then by *De Moivre's* theorem, $z^5=\cos 5\theta+i\sin 5\theta$. But by the Binomial theorem, $z^5=\sum_{k=0}^{5}\binom{5}{k}i^k\sin^k\theta\cos^{5-k}\theta$. Equating real parts,

$$\cos 5\theta=\cos^5\theta-10\sin^2\theta\cos^3\theta+5\sin^4\theta\cos\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta.$$

(a) Let $\cos\theta=x$. Then $\cos 5\theta=1\Leftrightarrow 16x^5-20x^3+5x=1$. Hence if θ is a solution of $\cos 5\theta=1$, $\cos\theta$ is a root of $16x^5-20x^3+5x-1=0$.

But $\cos 5\theta=1\Rightarrow 5\theta=0+2\pi n\Rightarrow\theta=\frac{2}{5}\pi n,\quad n=0,\pm1,\pm2,K$

Since the period of the function $\cos t$ is 2π, this formula gives under $n=0,1,2,3,4$ all the values of $\cos\theta$, namely

$$\cos 0=1,\cos\frac{2}{5}\pi,\cos\frac{4}{5}\pi=-\cos\frac{\pi}{5},\cos\frac{6}{5}\pi=-\cos\frac{\pi}{5},\cos\frac{8}{5}\pi=\cos\frac{2}{5}\pi.$$

But $\cos\theta=x$ and hence $16x^5-20x^3+5x-1=0$ has roots

$$1,\cos\frac{2}{5}\pi,\cos\frac{2}{5}\pi,\cos\frac{4}{5}\pi,\cos\frac{4}{5}\pi.$$

Let $\cos\frac{2}{5}\pi=a$, then $\cos\frac{4}{5}\pi=2\cos^2\frac{2}{5}\pi-1=2a^2-1$. The sum of roots of

$16x^5-20x^3+5x-1=0$ is $1+2a+2(2a^2-1)=0\Rightarrow 4a^2+2a-1=0\Rightarrow a=\frac{-1\pm\sqrt{5}}{4}$.

But

$a=\cos\frac{2}{5}\pi>0 \Rightarrow \cos\frac{2}{5}\pi=\frac{-1+\sqrt{5}}{4}$, and $\cos\frac{4}{5}\pi=2\left(\frac{-1+\sqrt{5}}{4}\right)^2-1=\frac{-1-\sqrt{5}}{4}=-\frac{1}{4}(1+\sqrt{5})$

.

(b) Let $\cos\theta=x$. Then $\cos 5\theta=\frac{1}{2} \Leftrightarrow 16x^5-20x^3+5x=\frac{1}{2}$. Hence if θ is a solution of $\cos 5\theta=\frac{1}{2}$, $\cos\theta$ is a root of $32x^5-40x^3+10x-1=0$.

But $\cos 5\theta=\frac{1}{2} \Rightarrow 5\theta=\pm\frac{\pi}{3}+2\pi n,$ n integral $\theta=\pm\frac{\pi}{15}+\frac{2}{5}\pi n,$ $n=0,\pm1,\pm2,K$ These values of θ give exactly five distinct values of $\cos\theta$, namely

$$\cos\frac{\pi}{15},\cos\frac{7}{15}\pi,\cos\frac{13}{15}\pi,\cos\frac{19}{15}\pi,\cos\frac{25}{15}\pi=\cos\frac{\pi}{3}=\frac{1}{2}.$$

(i) The sum of roots $32x^5-40x^3+10x-1=0$ is zero, hence

$$\cos\frac{\pi}{15}+\cos\frac{7}{15}\pi+\cos\frac{13}{15}\pi+\cos\frac{19}{15}\pi=-\frac{1}{2},$$

(ii) The product of roots is $\frac{1}{32}$, hence $\cos\frac{\pi}{15}\cos\frac{7}{15}\pi\cos\frac{13}{15}\pi\cos\frac{19}{15}\pi\cdot\frac{1}{2}=\frac{1}{32}$,

$$\cos\frac{\pi}{15}\cos\frac{7}{15}\pi\cos\frac{13}{15}\pi\cos\frac{19}{15}\pi=\frac{1}{16}.$$

Exercise 4.5

1 *Solution*

Let $\dfrac{2x+10}{(x-1)(x+3)}=\dfrac{c_1}{x-1}+\dfrac{c_2}{x+3}$. Then $2x+10=c_1(x+3)+c_2(x-1)$. Putting $x=1$ gives $c_1=3$, while $x=-3$ gives $c_2=-1$.

Hence $\dfrac{2x+10}{(x-1)(x+3)}=\dfrac{3}{x-1}-\dfrac{1}{x+3}$.

2 *Solution*

Using the quadratic formula, we get $2x^2+5x+3=2\left(x+\dfrac{3}{2}\right)(x+1)=(2x+3)(x+1)$.

Let $\dfrac{4x+5}{(2x+3)(x+1)}=\dfrac{c_1}{2x+3}+\dfrac{c_2}{x+1}$. Then $4x+5=c_1(x+1)+c_2(2x+3)$. Putting $x=-1$ gives $c_2=1$, while $x=-3/2$ gives $c_1=2$.

Hence $\dfrac{4x+5}{2x^2+5x+3}=\dfrac{2}{2x+3}+\dfrac{1}{x+1}$.

3 *Solution*

Using the quadratic formula, we get $2x^2-5x+2=2(x-1/2)(x-2)=(2x-1)(x-2)$.

Let $\dfrac{6}{(2x-1)(x-2)}=\dfrac{c_1}{2x-1}+\dfrac{c_2}{x-2}$. Then $6=c_1(x-2)+c_2(2x-1)$. Putting $x=2$ gives $c_2=2$, while $x=1/2$ gives $c_1=-4$.

Hence $\dfrac{6}{2x^2-5x+2}=\dfrac{-4}{2x-1}+\dfrac{2}{x-2}$.

4 *Solution*

By division, $\dfrac{x^2+x+2}{x(x+1)}=\dfrac{x^2+x+2}{x^2+x}=1+\dfrac{2}{x^2+x}=1+\dfrac{2}{x(x+1)}$.

Let $\dfrac{2}{x(x+1)}=\dfrac{c_1}{x}+\dfrac{c_2}{x+1}$. Then $2=c_1(x+1)+c_2x$. Putting $x=0$ gives $c_1=2$, while $x=-1$ gives $c_2=-2$.

Hence $\dfrac{x^2+x+2}{x(x+1)}=1+\dfrac{2}{x}-\dfrac{2}{x+1}$.

5 *Solution*

Let $\dfrac{2x+4}{(x-2)(x^2+4)}=\dfrac{c_1}{x-2}+\dfrac{ax+b}{x^2+4}$. Then $2x+4=c_1(x^2+4)+(ax+b)(x-2)$.

Putting $x=2$ gives $c_1=1$.

Equate coefficients of x^2: $0=c_1+a \Rightarrow a=-1$.

Put $x=0$: then $4=4c_1-2b \Rightarrow b=0$.

Hence $\dfrac{2x+4}{(x-2)(x^2+4)}=\dfrac{1}{(x-2)}-\dfrac{x}{(x^2+4)}$.

6 *Solution*

Let $\dfrac{3x^2-3x+2}{(2x-1)(x^2+1)}=\dfrac{c_1}{2x-1}+\dfrac{ax+b}{x^2+1}$. Then

$3x^2-3x+2=c_1(x^2+1)+(ax+b)(2x-1)$. Putting $x=1/2$ gives $c_1=1$.

Equate coefficients of x^2: $3=c_1+2a \Rightarrow a=1$.

Put $x=0$: then $2=c_1-b \Rightarrow b=-1$.

Hence $\dfrac{3x^2-3x+2}{(2x-1)(x^2+1)}=\dfrac{1}{2x-1}+\dfrac{x-1}{x^2+1}$.

7 *Solution*

It is necessary to perform the division transformation before seeking partial fractions.

By division,

$$\frac{x^3+2x^2+6x+10}{(x+1)(x^2+4)}=\frac{x^3+2x^2+6x+10}{x^3+x^2+4x+4}=1+\frac{x^2+2x+6}{x^3+x^2+4x+4}=1+\frac{x^2+2x+6}{(x+1)(x^2+4)}.$$

Let $\dfrac{x^2+2x+6}{(x+1)(x^2+4)}=\dfrac{c_1}{x+1}+\dfrac{ax+b}{x^2+4}$. Then $x^2+2x+6=c_1(x^2+4)+(ax+b)(x+1)$.

Putting $x=-1$ gives $c_1=1$.

Equate coefficients of x^2: $1=c_1+a \Rightarrow a=0$.

Put $x=0$: then $6=4c_1+b \Rightarrow b=2$.

Hence $\dfrac{x^3+2x^2+6x+10}{(x+1)(x^2+4)}=1+\dfrac{1}{x+1}+\dfrac{2}{x^2+4}$.

8 *Solution*

Let $\dfrac{5-x}{(2x+3)(x^2+1)}=\dfrac{c_1}{2x+3}+\dfrac{ax+b}{x^2+1}$. Then $5-x=c_1(x^2+1)+(ax+b)(2x+3)$.

Putting $x=-3/2$ gives $c_1=2$.

Equate coefficients of x^2: $\quad 0=c_1+2a \Rightarrow a=-1$.

Put $x=0$: then $5=c_1+3b \Rightarrow b=1$.

Hence $\dfrac{5-x}{(2x+3)(x^2+1)}=\dfrac{2}{2x+3}+\dfrac{1-x}{x^2+1}$.

9 *Solution*

Let $\dfrac{x^2+7}{(x^2+1)(x^2+4)}=\dfrac{ax+b}{x^2+1}+\dfrac{cx+d}{x^2+4}$. Then

$x^2+7=(ax+b)(x^2+4)+(cx+d)(x^2+1)$.

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^3: & 0=a+c \\ \text{Equate coefficients of } x\ \ : & 0=4a+c\end{array}\right\} \Rightarrow a=0, c=0.$$

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^2: & 1=b+d \\ \text{Equate constant terms} \quad : & 7=4b+d\end{array}\right\} \Rightarrow b=2, d=-1.$$

Hence $\dfrac{x^2+7}{(x^2+1)(x^2+4)}=\dfrac{2}{x^2+1}-\dfrac{1}{x^2+4}$.

10 *Solution*

Let $\dfrac{3x}{(x^2+1)(x^2+4)}=\dfrac{ax+b}{x^2+1}+\dfrac{cx+d}{x^2+4}$. Then $3x=(ax+b)(x^2+4)+(cx+d)(x^2+1)$.

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^3: & 0=a+c \\ \text{Equate coefficients of } x\ \ : & 3=4a+c\end{array}\right\} \Rightarrow a=1, c=-1.$$

$$\left.\begin{array}{ll}\text{Equate coefficients of } x^2: & 0=b+d \\ \text{Equate constant terms} \quad : & 0=4b+d\end{array}\right\} \Rightarrow b=0, d=0.$$

Hence $\dfrac{3x}{(x^2+1)(x^2+4)} = \dfrac{x}{x^2+1} - \dfrac{x}{x^2+4}$.

Diagnostic test 4

1 *Solution*

(i) In order to solve the polynomial equation $P(x)=x^4-4x^2+3=0$, denote x^2 as t and use the quadratic formula. Then $x^2=1$ or $x^2=3$. Hence $P(x)=(x^2-1)(x^2-3)$.

(a) Irreducible factors of $P(x)$ over **Q** are $P(x)=(x-1)(x+1)(x^2-3)$. Each linear factor gives rise to a zero of $P(x)$. Hence the zeros of $P(x)$ over **Q** are ± 1.

(b,c) Irreducible factors of $P(x)$ over **R** and over **C** are

$P(x)=(x-1)(x+1)(x-\sqrt{3})(x+\sqrt{3})$. Hence the zeros of $P(x)$ over **R** and over **C** are $\pm 1, \pm\sqrt{3}$.

(ii) Use the quadratic formula, then $P(x)=x^4-2x^2-3=0 \Rightarrow x^2=-1$ or $x^2=3$.

Hence $P(x)=(x^2+1)(x^2-3)$.

(a) $P(x)$ has no linear factors over **Q** and $P(x)=0$ has no solutions in the field of rational numbers. Hence $P(x)$ has no zeros over **Q**.

(b) Irreducible factors of $P(x)$ over **R** are $P(x)=(x^2+1)(x-\sqrt{3})(x+\sqrt{3})$. Each linear factor gives rise to a zero of $P(x)$. Hence the zeros of $P(x)$ over **R** are $\pm\sqrt{3}$.

(c) Irreducible factors of $P(x)$ over **C** are $P(x)=(x-i)(x+i)(x-\sqrt{3})(x+\sqrt{3})$.

Hence the zeros of $P(x)$ over **C** are $\pm i, \pm\sqrt{3}$.

2 *Solution*

(a,b) If α is a rational zero of $P(x)$, then α is a divisor of the constant term. Hence the only rational zeros of $P(x)$ are $\pm 1, \pm 5, \pm 10$. By inspection,

$P(1)=0$ and $P(-5)=0$. Hence $(x-1)$ and $(x+5)$ are the factors of $P(x)$. Dividing $P(x)$ by $(x-1)(x+5)=x^2+4x-5$ we obtain

$P(x)=(x-1)(x+5)(x^2+2)$, and these are irreducible factors over **Q** and **R**.

(c) Irreducible factors of $P(x)$ over **C** are $P(x)=(x-1)(x+5)(x-\sqrt{2}\,i)(x+\sqrt{2}\,i)$.

3 *Solution*

$P(x) = 4x^3 + 15x^2 + 12x - 4,$

$P'(x) = 12x^2 + 30x + 12, \Rightarrow P'(-2) = 0, \quad P(-2) = 0.$

Hence -2 is a double zero of $P(x)$ and $P(x) = 4(x+2)^2(x+k)$ for some constant k, as $P(x)$ is a polynomial of degree 3 with the leading coefficient 4. Then $P(0) = -4 \Rightarrow k = -1/4$ and $P(x) = (x+2)^2(4x-1)$. The zeros of $P(x)$ are $-2, -2, 1/4$.

4 *Solution*

$P(x) = 2x^3 - x^2 - 6x + 3$. All rational zeros of $P(x)$ have the form p/q, where p and q

are integer divisors of 3 and 2 respectively. Hence the only possible rational zeros of $P(x)$

are $\pm 1, \pm 3, \pm 3/2$. But of these, only $1/2$ satisfies $P(x) = 0$. Hence $(2x-1)$ is a factor of $P(x)$.. By polynomial division,

$P(x) = (2x-1)(x^3-3) = (2x-1)(x-\sqrt{3})(x+\sqrt{3})$, and these are irreducible factors of $P(x)$ over the real numbers. Each linear factor gives rise to a zero of $P(x)$. Hence the zeros of $P(x)$ are $1/2, \ \pm\sqrt{3}$.

5 *Solution*

(a) $x - i$ is a linear factor. Hence we can use the remainder theorem, and the remainder is

$P(i) = -i - 2 - 1 = -3 - i$.

(b) $P(x) = x^3 + 2x^2 - 1$ and $D(x) = x^2 + 1$ are polynomials over **Q**. By the division transformation, $P(x) \equiv (x^2+1)Q(x) + R(x)$, where $Q(x)$ and $R(x)$ are polynomials over **Q**, such that $\deg R < \deg D = 2$. Thus $P(x) \equiv (x^2+1)Q(x) + ax + b, \quad a, b$ rational, and this equation is true for all $x \in \mathbf{C}$. Then

$P(i) = -i - 2 - 1 = -3 - i \Rightarrow -3 - i = ai + b$. But a and b are real $\Rightarrow a = -1, b = -3$.

Hence the remainder is $ax + b = -x - 3$.

6 *Solution*

$P(x)=x^4+ax^2+bx$. By the division transformation, $P(x)=(x^2+1)Q(x)+x+2$.

Then

$$\left.\begin{array}{l} P(i)=1-a+bi \Rightarrow 1-a+bi=i+2 \\ P(-i)=1-a-bi \Rightarrow 1-a-bi=-i+2 \end{array}\right\} \Rightarrow 1-a=2, b=1.$$

Hence $a=-1, b=1$.

7 *Solution*

$P(x)$ has real coefficients. Hence $P(1-i)=0 \Rightarrow P(1+i)=0$ and then

$[x-(1-i)][x-(1+i)]=x^2-2x+2$ is a factor of $P(x)$. By polynomial division,

$P(x)=(x^2-2x+2)(x^2-3)$. Hence $P(x)=(x^2-2x+2)(x-\sqrt{3})(x+\sqrt{3})$, this is the factorisation of $P(x)$ into irreducible factors over **R**, and $P(x)$ has zeros $1\pm i$, $\pm\sqrt{3}$.

8 *Solution*

$P(x)=x^4+ax^2+6$, as $P(x)$ is an even monic polynomial of degree 4. Then

$P(\sqrt{2})=0 \Rightarrow 4+2a+6=0$. Hence $a=-5$ and $P(x)=x^4-5x^2+6$.

$P(x)$ is even. Hence $P(\sqrt{2})=0 \Rightarrow P(-\sqrt{2})=0$ and then $(x-\sqrt{2})(x+\sqrt{2})=x^2-2$

is a

factor of $P(x)$. By inspection, $P(x)=(x^2-2)(x^2-3)$. So the irreducible factors of

$P(x)$

are $P(x)=(x-\sqrt{2})(x+\sqrt{2})(x-\sqrt{3})(x+\sqrt{3})$.

9 *Solution*

Let the roots of $P(x)=x^3-3x^2+ax+8$ be $c-b, c, c+b$. Then sum of roots is equal to 3, hence $3c=3 \Rightarrow c=1$. Product of roots is $-8 \Rightarrow 1-b^2=-8 \Rightarrow b=3$ or $b=-3$ (that gives the same roots of $P(x)$). Hence the roots of $P(x)$ are $-2, 1, 4$. Therefore $\sum \alpha\beta=-2-8+4=-6$, hence $a=-6$.

10 *Solution*

(a) $2\alpha, 2\beta$ and 2γ satisfy $\left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^2 - 2\left(\frac{x}{2}\right) - 3 = 0$. Hence the required equation is

$x^3 + 2x^2 - 8x - 24 = 0$.

(b) $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$ satisfy $(2x)^3 + (2x)^2 - 2(2x) - 3 = 0$. Hence the required equation is

$8x^3 + 4x^2 - 4x - 3 = 0$.

(c) $\alpha - 2, \beta - 2$ and $\gamma - 2$ satisfy $(x+2)^3 + (x+2)^2 - 2(x+2) - 3 = 0$. Hence the required equation is $x^3 + 7x^2 + 14x + 5 = 0$.

(d) $\alpha + 2, \beta + 2$ and $\gamma + 2$ satisfy $(x-2)^3 + (x-2)^2 - 2(x-2) - 3 = 0$. Hence the required equation is $x^3 - 5x^2 + 6x - 3 = 0$.

11 *Solution*

(a) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ satisfy $\left(\frac{1}{x}\right)^3 + q\left(\frac{1}{x}\right) + r = 0$. Hence the required equation is

$rx^3 + qx^2 + 1 = 0$.

(b) α^2, β^2 and γ^2 satisfy $\left(x^{1/2}\right)^3 + qx^{1/2} + r = 0$. Rearrangement gives

$x^{1/2}(x+q) = -r$. Squaring and simplifying, the required equation is

$x^3 + 2qx^2 + q^2x - r^2 = 0$.

12 *Solution*

(a) $\alpha + \beta + \gamma = 0$, as the coefficient of x^2 is zero.

(b) α^2, β^2 and γ^2 satisfy $\left(x^{1/2}\right)^3 + 2x^{1/2} + 1 = 0$. Rearrangement gives

$x^{1/2}(x+2) = -1$. Squaring and simplifying, $x^3 + 4x^2 + 4x - 1 = 0$. Hence the sum of the roots of this equation

$\alpha^2 + \beta^2 + \gamma^2 = -4$.

(c) $\alpha^3+2\alpha+1=0$ (since α,β,γ are roots of the given equation),

$\beta^3+2\beta+1=0,$

$\gamma^3+2\gamma+1=0,$

hence $\left(\alpha^3+\beta^3+\gamma^3\right)+2(\alpha+\beta+\gamma)+3=0$. But from (a) $\alpha+\beta+\gamma=0$, and

$\alpha^3+\beta^3+\gamma^3=-3$.

(d) From (b) α^2, β^2 and γ^2 satisfy $x^3+4x^2+4x-1=0$. Hence α^4, β^4 and γ^4 satisfy $\left(x^{1/2}\right)^3+4\left(x^{1/2}\right)^2+4x^{1/2}-1=0$. Rearrangement gives $x^{1/2}(x+4)=1-4x$.

Squaring and simplifying, $x^3-8x^2+24x-1=0$. Hence the sum of the roots of this equation

$\alpha^4+\beta^4+\gamma^4=8$.

13 *Solution*

$P(x)$ has symmetric coefficients, hence it can be converted to quadratic equation in

$\left(x+\frac{1}{x}\right)$

$$P(x)=3x^4-4x^3-14x^2-4x+3=x^2\left\{3x^2+\frac{3}{x^2}-4x-\frac{4}{x}-14\right\}.$$

Using $\left(x+\frac{1}{x}\right)^2=x^2+2+\frac{1}{x^2}$, we get $P(x)=x^2\left\{3\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)-20\right\}$. Since 0 is not a zero of $P(x)$, the solutions of $P(x)$ are the solutions of

$3\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)-20=0$. By factorising this quadratic

$$P(x)=x^2\left\{3\left(x+\frac{1}{x}\right)-10\right\}\left(x+\frac{1}{x}+2\right)=\left(3x^2-10x+3\right)\left(x^2+2x+1\right).$$

Hence

$P(x)=0 \Rightarrow 3x^2-10x+3=0$ or $x^2+2x+1=0,$ and

$$x=\frac{5\pm\sqrt{16}}{3} \quad \text{or} \quad x=-1.$$

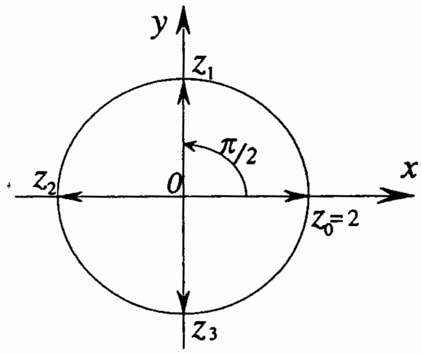
So the roots of $P(x)=0$ are $-1,-1,1/3,3,$ and from

$P(x)=\left(3x^2-10x+3\right)\left(x^2+2x-1\right)$ it follows that the factorisation of $P(x)$ over $\mathbf{R}$ is

$P(x)=(3x-1)(x-3)(x+1)^2$.

14 *Solution*

$z^5-16z=0 \Rightarrow z(z^4-16)=0$. Hence $z=0$ or z is a complex root of 16. Clearly, one such root is 2, as $2^4=16$. The other three roots are equally spaced by $\frac{\pi}{2}$ around a circle of radius 2 and center $(0,0)$ in the Argand diagram. The fourth roots of 16 are ± 2 and $\pm 2i$. Hence z^5-16z has the roots $0, \pm 2, \pm 2i$.



15 *Solution*

Let $z=\cos\theta+i\sin\theta$. Then by *De Moivre's* theorem, $z^3=\cos 3\theta+i\sin 3\theta$. But by the Binomial theorem, $z^3=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$. Equating real and imaginary parts, $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$ and $\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$.

Dividing one by another, $\tan 3\theta=\dfrac{3\cos^2\theta\sin\theta-\sin^3\theta}{\cos^3\theta-3\cos\theta\sin^2\theta}=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$.

Furthermore, it is clearly that the equation $P(x)=x^3-3x^2-3x+1=0$ has the integer root -1. By polynomial division, $x^3-3x^2-3x+1=(x+1)(x^2-4x+1)$. Using the quadratic formula, the roots of the equation are $-1, 2\pm\sqrt{3}$.

If $\theta=\dfrac{\pi}{12}$, then $3\theta=\dfrac{\pi}{4}$ and $\tan 3\theta=1$.

Hence $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta} \Rightarrow \tan^3\theta-3\tan^2\theta-3\tan\theta+1=0$. Let $x=\tan\theta$, then

$\tan^3\theta-3\tan^2\theta-3\tan\theta+1=0 \Leftrightarrow x^3-3x^2-3x+1=0$. But $0<\dfrac{\pi}{12}<\dfrac{\pi}{4}$.

Then $0 < \tan\frac{\pi}{12} < 1 \Rightarrow \tan\frac{\pi}{12} = 2 - \sqrt{3}$.

Now $\frac{5\pi}{12} = \frac{\pi}{2} - \frac{\pi}{12} \Rightarrow \tan\frac{5\pi}{12} = \frac{1}{\tan\pi/12} = \frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})} = 2+\sqrt{3}$.

16 *Solution*

Let $z = \cos\theta + i\sin\theta$. Then by *De Moivre's* theorem, $z^5 = \cos 5\theta + i\sin 5\theta$. But by the Binomial theorem, $z^5 = \sum_{k=o}^{5}\binom{5}{k} i^k \sin^k\theta\cos^{5-k}\theta$. Equating real parts,

$\cos 5\theta = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

Furthermore, $16x^5 - 20x^3 + 5x = 0 \Rightarrow x(16x^4 - 20x^2 + 5) = 0$. Hence

$x = 0$ or $16x^4 - 20x^2 + 5 = 0$.

To solve the last equation use the quadratic formula in x^2, then $x^2 = \frac{5\pm\sqrt{5}}{8}$.

So the roots are $0, \pm\sqrt{\frac{5+\sqrt{5}}{8}}, \pm\sqrt{\frac{5-\sqrt{5}}{8}}$.

Let $x = \cos\theta$. Then

$$\cos 5\theta = 0 \Leftrightarrow 16x^5 - 20x^3 + 5x = 0. \text{ If } \theta = \frac{\pi}{10} \text{ or } \theta = \frac{3\pi}{10} \Rightarrow \cos 5\theta = 0.$$

But $0 < \frac{\pi}{10} < \frac{3\pi}{10} < \frac{\pi}{2}$, hence

$$\cos\frac{\pi}{10} > \cos\frac{3\pi}{10} > 0 \Rightarrow \cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}, \cos\frac{3\pi}{10} = \sqrt{\frac{5-\sqrt{5}}{8}}.$$

17 *Solution*

Using the quadratic formula, $x^2 - x - 6 = (x-3)(x+2)$.

Let $\frac{3x-4}{(x-3)(x+2)} = \frac{c_1}{x-3} + \frac{c_2}{x+2}$. Then $3x - 4 = c_1(x+2) + c_2(x-3)$. Putting $x = -2$ gives

$c_2 = 2$, while $x = 3$ gives $c_1 = 1$. Hence $\frac{3x-4}{x^2-x-6} = \frac{1}{x-3} + \frac{2}{x+2}$.

18 *Solution*

Let $\dfrac{3x^2-6x+10}{(x-4)(x^2+1)}=\dfrac{c_1}{x-4}+\dfrac{ax+b}{x^2+1}$. Then $3x^2-6x+10=c_1(x^2+1)+(ax+b)(x-4)$.

Put $x=4$: then $c_1=2$.

Equate coefficients of x^2: $3=c_1+a \Rightarrow a=1$.

Equate constant terms : $10=c_1-4b \Rightarrow b=-2$.

Hence $\dfrac{3x^2-6x+10}{(x-4)(x^2+1)}=\dfrac{2}{x-4}+\dfrac{x-2}{x^2+1}$.

Further questions 4

1 Solution

Let $P(x)=2x^3-13x-\sqrt{7}$, then $P(\sqrt{7})=14\sqrt{7}-13\sqrt{7}-\sqrt{7}=0$. Hence $x-\sqrt{7}$ is a factor of $P(x)$. By polynomial division, $P(x)=(x-\sqrt{7})(2x^2+2\sqrt{7}x+1)$.

Factorising $2x^2+2\sqrt{7}x+1$,

$P(x)=(x-\sqrt{7})\left(x-\dfrac{\sqrt{5}-\sqrt{7}}{2}\right)(2x+\sqrt{5}+\sqrt{7})$. Hence the roots of the equation

$P(x)=0$

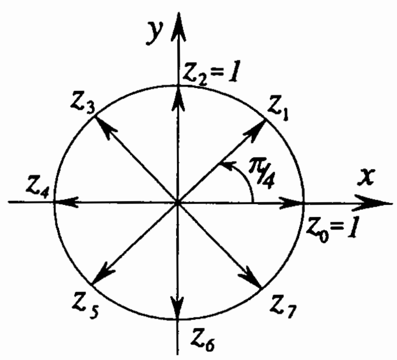
are $\sqrt{7},\ \dfrac{-\sqrt{7}\pm\sqrt{5}}{2}$.

2 Solution

It is clear that $x^8-1=(x^2)^4-1=(x^2-1)(x^6+x^4+x^2+1)$. Hence

$x^8-1=0 \Leftrightarrow x^2-1=0$ or $P(x)=x^6+x^4+x^2+1=0$. So the zeros of $P(x)$ are the solutions of $x^8=1,\ x\neq\pm1$. Clearly

$z^8=1 \Rightarrow z$ is a complex eighth root of unity. These roots are equally spaced around the unit circle in the Argand diagram, the angular spacing being $\dfrac{2\pi}{8}=\dfrac{\pi}{4}$. The non-real eighth roots of unity are



z_1 and $\overline{z_1}$, where $z_1=\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}$,

z_2 and $\overline{z_2}$, where $z_2=i$,

z_3 and $\overline{z_3}$, where $z_3=-\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}$.

Hence $P(x)=0$ has the roots

$\dfrac{1}{\sqrt{2}}(1\pm i),\ \pm i,\ \dfrac{1}{\sqrt{2}}(-1\pm i)$, and

$$P(x)=(x-z_1)(x-\overline{z_1})(x-z_2)(x-\overline{z_2})(x-z_3)(x-\overline{z_3})$$
$$=(x^2-2\operatorname{Re}z_1x+|z_1|^2)(x^2-2\operatorname{Re}z_2x+|z_2|^2)(x^2-2\operatorname{Re}z_3x+|z_3|^2).$$

Hence $P(x)=(x^2-x\sqrt{2}+1)(x^2+1)(x^2+x\sqrt{2}+1)$ is a full factorisation of $P(x)$ over **R**.

3 Solution

The quatic equation $P(x)=5x^4-11x^3+16x^2-11x+5=0$ has symmetric coefficients and so it can be converted to quadratic equation in $\left(x\pm\frac{1}{x}\right)$.

$P(x)=x^2\left\{5(x^2+1/x^2)-11(x+1/x)+16\right\}=x^2\left\{5(x+1/x)^2-11(x+1/x)+6\right\}$.

Since 0 is a not a zero of $P(x)$, the solutions of $P(x)$ are the solutions of $5(x+1/x)^2-11(x+1/x)+6=0$. By factorising this quadratic,

$P(x)=x^2\left\{x+\frac{1}{x}-1\right\}\left\{5+\left(x+\frac{1}{x}\right)-6\right\}=(x^2-x+1)(5x^2-6x+5)$. Hence

$P(x)=0 \Leftrightarrow x^2-x+1=0$ or $5x^2-6x+5=0$, and

$$x=\frac{1\pm\sqrt{3}\,i}{2} \quad \text{or} \quad x=\frac{3\pm 4i}{5}.$$

So the roots of $P(x)=0$ are $\frac{1}{2}(1\pm\sqrt{3}\,i)$, $\frac{1}{5}(3\pm 4i)$. Since these zeros are non-real, the full factorisation of $P(x)$ over **R** is $P(x)=(x^2-x+1)(5x^2-6x+5)$.

4 Solution

Let $z=a+ib, b\neq 0$ and $P(a+ib)=0 \Rightarrow P\left(\bar{z}\right)=P(a-ib)=0$. If $a+ib$ is a double zero, then

$a-ib$ is also double zero too. Since $P(x)$ is a monic polynomial of degree four,

$P(x)=(x-z)^2(x-\bar{z})^2=\left\{(x-z)(x-\bar{z})\right\}^2=\left(x^2-2\operatorname{Re}zx+|z|^2\right)^2=x^2-2ax+(a^2+b^2)$

$=x^4-4ax^3+(6a^2+b^2)x^2+4a(a^2+b^2)x+(a^2+b^2)^2$. But $P(x)=x^4-8x^3+30x^2-56x+4$

Equate coefficients of x^3: $-4a=-8 \Rightarrow a=2$.

Equate constant terms : $(a^2+b^2)^2=49 \Rightarrow b^2=3$.

Hence the roots of $P(x)$ are $2\pm\sqrt{3}\,i$, and the irreducible factors of $P(x)$ over **R** are

$$P(x)=\left\{x^2-2ax+(a^2+b^2)\right\}^2=(x^2-4x+7)^2.$$

5 Solution

Let $P(z)=z^4+3z^2-6z+10$, then

$$P(1+i)=(1+4i+6i^2+4i^3+i^4)+3(1+2i+i^2)-6(1+i)+10=-4+6i-6-6i+10=0$$

.

$P(z)$ has real coefficients, hence $P(1+i)=0 \Rightarrow P\left(\overline{1+i}\right)=P(1-i)=0$.

Then $\{z-(1+i)\}\left\{z-\left(\overline{1+i}\right)\right\}=(z^2-2z+2)$ is a factor of $P(z)$. By polynomial division,

$P(z)=(z^2-2z+2)(z^2+2z+5)$. Using the quadratic formula, $z^2+2z+5=0 \Rightarrow$ $z=-1\pm 2i$. Hence the roots of $P(z)=0$ are $1\pm i, -1\pm 2i$.

6 Solution

Let $x=y+\dfrac{1}{y} \Rightarrow y^2+xy+1=0 \Rightarrow y=\dfrac{x+\sqrt{x^2-4}}{2}$ or $y=\dfrac{x-\sqrt{x^2-4}}{2}$.

Then $\alpha+\dfrac{1}{\alpha}, \beta+\dfrac{1}{\beta}$ and $\gamma+\dfrac{1}{\gamma}$ satisfy

$$\left\{\frac{x\pm\left(x^2-4\right)^{1/2}}{2}\right\}^3+3\left\{\frac{x\pm\left(x^2-4\right)^{1/2}}{2}\right\}+2=0.$$

Rearrangement gives $\pm\left(x^2-4\right)^{1/2}\left(4x^2+8\right)=-4x^3-16$. Squaring and simplifying, we get

$2x^3+3x^2+8=0$.

7 Solution

The sum of roots of the equation $x^4-px^3+qx^2-pqx+1=0$ is equal to

$\alpha+\beta+\gamma+\delta=p$.

Then

$t := (\alpha+\beta+\gamma)(\alpha+\beta+\delta)(\alpha+\gamma+\delta)(\beta+\gamma+\delta) = (p-\delta)(p-\gamma)(p-\beta)(p-\alpha)$.

Expanding,

$t = p^4 - p^3(\alpha+\beta+\gamma+\delta) + p^2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) - p(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)+\alpha\beta\gamma\delta$

.

But $\sum\alpha = p,\ \sum\alpha\beta = q,\ \sum\alpha\beta\gamma = pq,\ \alpha\beta\gamma\delta = 1$. Hence

$t = p^4 - p^4 + p^2q - p^2q + 1 = 1$.

8 Solution

Let $P(x) = x^4 - px^3 + qx^2 - rx + s$. Then

$P(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = \left\{x^2-(\alpha+\beta)x+\alpha\beta\right\}\left\{x^2-(\gamma+\delta)x+\gamma\delta\right\}$

$= x^4 - (\alpha+\beta+\gamma+\delta)x^3 + \left\{(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta\right\}x^2 - \left\{\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)\right\}x + \alpha\beta\gamma\delta.$

(a) Equate constant terms: $\alpha\beta\gamma\delta = s$. But $\alpha\beta = \gamma\delta \Rightarrow (\alpha\beta)^2 = s$. At the same time

the

coefficient of x:

$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = \alpha\beta(\alpha+\beta+\gamma+\delta) = \alpha\beta\cdot p$, as $\alpha+\beta+\gamma+\delta = p$.

Equate coefficients of x: $\alpha\beta\cdot p = r \Rightarrow r^2 = (\alpha\beta)^2p^2 \Rightarrow r^2 = sp^2$.

(b) Equate coefficients of x^3: $\alpha+\beta+\gamma+\delta = p$. But $\alpha+\beta = \gamma+\delta \Rightarrow \alpha+\beta = p/2$.

Equate coefficients of x^2: $(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta = q \Rightarrow \alpha\beta+\gamma\delta = q-(p/2)^2$.

Equate coefficients of x: $\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) = r \Rightarrow (\alpha\beta+\gamma\delta)(\alpha+\beta) = r \Rightarrow$

$$\left\{q-\left(\frac{p}{2}\right)^2\right\}\frac{p}{2} = r \Rightarrow p^3 - 4pq + 8r = 0.$$

9 Solution

Let $Q(z) = z^{n-1} + z^{n-2} + \text{K} + z + 1$. Then $Q(1) = n$.

Furthermore, $z^n - 1 = (z-1)(z^{n-1}+z^{n-2}+\text{K}+z+1)$. Hence the roots of

$Q(z) = 0$ are $z_1, z_2, \text{K}, z_{n-1}$.

Therefore $Q(z) = (z-z_1)(z-z_2)\text{K}(z-z_{n-1})$ and $Q(1) = (1-z_1)(1-z_2)\text{K}(1-z_{n-1})$.

But

$Q(1) = n \ \Rightarrow\ (1-z_1)(1-z_2)\text{K}(1-z_{n-1}) = n$.

10 Solution

Let $P(x)=x^3+3px+3qx+r$. A double root of $P(x)=0$ must be a single root of $P'(x)=0$.

Then

$P'(x)=3x^2+6px+3q=0 \Rightarrow x=-p\pm\sqrt{p^2-q}$. Let $\varepsilon=\pm 1$, $k=\sqrt{p^2-q}$, then $x=-p+\varepsilon k$.

Let us calculate $P(-p+\varepsilon k)$: $x^3=(-p+\varepsilon k)^3=-4p^3+3pq+(4p^2-q)\varepsilon k$,

$$3px^2=3p(-p+\varepsilon k)=6p^3-3pq-6p^2\varepsilon k,$$
$$3qx=3q(-p+\varepsilon k)=-3pq+3q\varepsilon k.$$

Hence $P(-p+\varepsilon k)=2p^3-3pq+\varepsilon k(-2p^2+2q)+r$. But it must be $P(-p+\varepsilon k)=0$.

Therefore, $\varepsilon k(-2p^2+2q)=-2p^3+3pq-r$. Squaring,

$$(p^2-q)(4p^4+4q^2-8p^2q)=4p^6+9p^2q^2+r^2-12p^4q+4p^3r-6pqr.$$

This is equivalent to

$$4(p^2-q)q^2+(p^2-q)(4p^4-8p^2q)=4p^6+9p^2q^2+r^2-12p^4q-2pqr+(p^2-q)4pr$$

Rearrangement gives

$$4(p^2-q)(q^2-pr)=-(p^2-q)(4p^4-8p^2q)+4p^6+9p^2q^2+r^2-12p^4q-2pqr.$$

Simplifying the right hand side of this identity,

$$4(p^2-q)(q^2-pr)=p^2q^2-2pqr+r^2.$$

Hence $4(p^2-q)(q^2-pr)=(pq-r)^2$.

11 Solution

Let $P(x)=x^n+px-q$. Then $P'(x)=nx^{n-1}+p$. Hence $P'(x)\Rightarrow x^{n-1}=-p/n$.

(Remember, that α is a double root of $P(x)=0$ if and only if $P(\alpha)=0$ and $P'(\alpha)=0$,

but $P''(\alpha)\neq 0$). Furthermore, $P(x)=0 \Leftrightarrow x(x^{n-1}+p)=q$. Substituting $x^{n-1}=-p/n$, we obtain

$x\left(\frac{-p}{n}+p\right)=q \;\Rightarrow\; x=\frac{nq}{(n-1)p}$. But $x^{n-1}=-p/n$.

Hence $\left\{\dfrac{nq}{(n-1)p}\right\}^{n-1} = \dfrac{-p}{n} \Rightarrow \left(\dfrac{p}{n}\right)^n + \left(\dfrac{q}{n-1}\right)^{n-1} = 0.$

12 Solution

Let $P(x) = \sum_{r=0}^{n} a_r x^{n-r} = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \mathrm{K} + a_{n-1}x + a_n$.

Then $P(x) = (x-1)(x-2)\,\mathrm{K}\,(x-n)$, as the roots of $P(x)$ are the first n positive integers.

Hence $a_1 = -\sum \alpha = -\sum_{k=1}^{n} k = -\frac{1}{2}n(n+1)$ and $a_n = \prod(-\alpha) = \prod_{k=1}^{n}(-k) = (-1)^n n!$.

At the same time

$a_2 = \sum \alpha\beta = \sum_{1 \le k < l \le n} kl$.

In order to calculate this last sum, we use the following identity

$\left(\sum_{k=1}^{n} a_k\right)^2 = \sum_{k=1}^{n} a_k^2 + 2 \sum_{1 \le k < l \le n} a_k a_l$, where $a_k \in \mathbf{C},\ k = 1, \mathrm{K}, n$.

Hence $a_2 = \dfrac{\left(\sum_{k=1}^{n} k\right)^2 - \sum_{k=1}^{n} k^2}{2}$. But $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$.

So $a_2 = \frac{1}{8}n^2(n+1)^2 - \frac{1}{12}n(n+1)(2n+1) = \frac{1}{24}n(n+1)\{3n(n+1) - 2(2n+1)\}$

$= \frac{1}{24}n(n+1)(3n^2 - n - 2) = \frac{1}{24}n(n+1)(n-1)(3n+2)$.