7SD Solutions Series

Worked Solutions to Popular Mathematics Texts

Suggested Worked Solutions to

"4 Unit Mathematics"

(Text book for the NSW HSC by D. Arnold and G. Arnold)

Specimen Papers 1 to 6

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7SD Solutions Series

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Solutions are to "4 Unit Mathematics"

[by D. Arnold and G. Arnold (1993), ISBN 0 340 54335 3]

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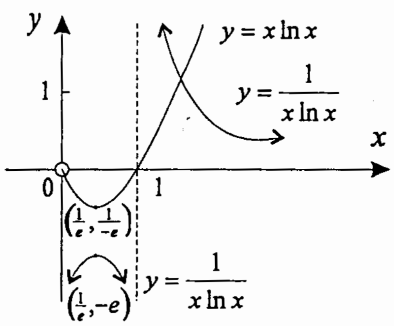
Specimen Paper 1.

1 Solution

1 (a)

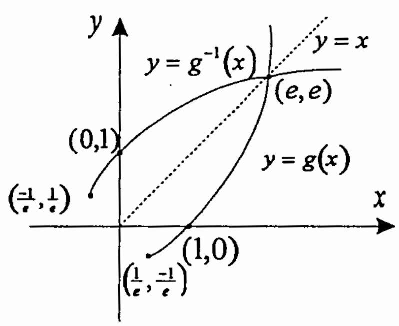


(i) Let us sketch the graph $y = f(x) = x\ln x$

Features of $y = f(x)$

- $y = 0$ at $x = 0$, $y = 0$ at $x = 1$
- $y \to +\infty$ as $x \to +\infty$
- $\dfrac{dy}{dx} = \ln x + 1$, $\dfrac{dy}{dx} = 0$ if $\ln x = 1 \Rightarrow x = e^{-1}$

  (stationary point - minimum), $f(e^{-1}) = -e^{-1}$
- Domain $\{x\colon\ x \ge 0\}$, Range $\{y\colon\ -e^{-1} \le y < +\infty\}$



(ii) Let us sketch the graph $y = h(x) = \dfrac{1}{x\ln x}$

Features of $y = h(x)$

- $y \to -\infty$ as $x \to 0^+$, $y \to -\infty$ as $x \to 1^-$

  $y \to +\infty$ as $x \to 1^+$, $y \to 0^+$ as $x \to +\infty$
-

$\dfrac{dy}{dx} = -\dfrac{\ln x + 1}{x^2\ln^2 x}$, $\dfrac{dy}{dx} = 0$ if $\ln x = -1 \Rightarrow x = e^{-1}$

(stationary point - maximum), $h(e^{-1}) = -e$

- Domain $\{x\colon\ 0 < x < 1$ or $1 < x < +\infty\}$

  Range $\{y\colon\ -\infty < y < -e$ or $y > 0\}$

(b)

(i) $y = g(x) = x\ln x$ for $x \ge \dfrac{1}{e}$

Features of $y = g(x)$

- end point: $g = -e^{-1}$ at $x = \dfrac{1}{e}$
- point of intersection with the x-axis: $g(x) = 0 \Rightarrow x = 1$
- point of intersection with the line $y = x$: $x\ln x = x \Rightarrow x = e$, $g(e) = e$

(ii) $y = g^{-1}(x)$ for $x \geq \frac{1}{e}$

Features of $y = g^{-1}(x)$

- end point: $x = -\frac{1}{e}$, $y = \frac{1}{e}$
- point of intersection with the x-axis: $x = 0$, $y = 1$
- point of intersection with the line $y = x$: $x = e$, $y = e$

(iii) We note that $\frac{d}{dx}\ln x = \frac{1}{x}$. Hence, integration by parts with $\ln x$ as the second function yields

$$\int_1^e x\ln x dx = \left[\frac{x^2}{2}\ln x\right]_1^e - \frac{1}{2}\int_1^e x^2 \frac{1}{x}dx = \frac{e^2}{2} - 0 - \frac{1}{2}\int_1^e x dx = \frac{e^2}{2} - \frac{1}{4}\left[x^2\right]_1^e = \frac{1}{4}(e^2+1).$$

The desired area S is equal to

$$S = 2\left(\int_0^e x dx - \int_1^e x\ln x dx\right) = 2\left(\frac{e^2}{2} - \frac{1}{4}(e^2+1)\right) = \frac{1}{2}(e^2-1).$$

2 Solution

(a)

$$\int \frac{e^{2x}}{1+e^x}dx = \int \frac{e^x de^x}{1+e^x} = \int \frac{u du}{1+u} = \int\left(1 - \frac{1}{1+u}\right)du = \int du - \int \frac{du}{1+u} = u - \ln|1+u| = e^x - \ln\left(1+e^x\right)$$

(b)

$$I \equiv \int \frac{\sin^3 x}{\cos^4 x}dx = \int \frac{\sin^2 x}{\cos^4 x}\left(-\cos x\right)' dx = -\int \frac{1-\cos^2 x}{\cos^4 x} d\cos x.$$

Using the substitution $\cos x = u$,

$$I = -\int \frac{1-u^2}{u^4}du = -\int \frac{du}{u^4} + \int \frac{du}{u^2} = \frac{1}{3u^3} - \frac{1}{u} = \frac{1}{3\cos^3 x} - \frac{1}{\cos x}.$$

(c)

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$$ Use the substitution $u = \tan\frac{x}{2}$;

$$du = \frac{1}{2}\frac{dx}{\cos^2\frac{x}{2}} \Rightarrow du = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow$$

$dx = \dfrac{2du}{1+u^2}$.

Now $\displaystyle\int_0^{\pi/2} \frac{dx}{4+5\sin x} = \int_0^1 \frac{1}{4+5\cdot\dfrac{2u}{1+u^2}}\cdot\frac{2du}{1+u^2} = \int_0^1 \frac{2du}{4u^2+10u+4} = \frac{1}{2}\int_0^1 \frac{du}{\left(u+\dfrac{5}{4}\right)^2 - \dfrac{9}{16}} =$

$$\frac{1}{2}\int_0^1 du\left\{\frac{1}{u+\dfrac{5}{4}-\dfrac{3}{4}} - \frac{1}{u+\dfrac{5}{4}+\dfrac{3}{4}}\right\}\frac{4}{6} = \frac{1}{3}\int_0^1 du\left\{\frac{1}{u+\dfrac{1}{2}} - \frac{1}{u+2}\right\} = \left[\frac{1}{3}\ln\frac{u+1/2}{u+2}\right]_0^1 =$$

$$\frac{1}{3}\left\{\ln\frac{1+1/2}{1+2} - \ln\frac{1/2}{2}\right\} = \frac{1}{3}\left(\ln\frac{1}{2} - \ln\frac{1}{4}\right) = \frac{1}{3}\ln 2.$$

(d) We use integration by parts.

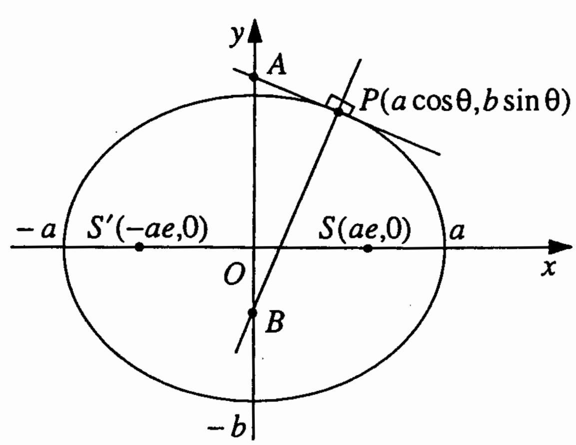
$$\int_0^1 \frac{\sin^{-1}x}{\sqrt{1+x}}dx = \int_0^1 \sin^{-1}x\left(2\sqrt{1+x}\right)' dx = \left[\sin^{-1}x\left(2\sqrt{1+x}\right)\right]_0^1 - \int_0^1 \left(\sin^{-1}x\right)' 2\sqrt{1+x}\,dx =$$

$$\frac{\pi}{2}\cdot 2\sqrt{2} - \int_0^1 \frac{2\sqrt{1+x}}{\sqrt{1-x^2}}dx = \sqrt{2}\pi - 2\int_0^1 \frac{dx}{\sqrt{1-x}} = \sqrt{2}\pi + \left[4\sqrt{1-x}\right]_0^1 = \sqrt{2}\pi - 4.$$

3 Solution

(a) The tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has gradient $\dfrac{b\sec\theta}{a\tan\theta}$. If P is an extremity in the first quadrant of a latus rectum, then $a\sec\theta = ae$. Thus $\sec\theta = e \Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{e^2-1}$. Since for the hyperbola $b^2 = a^2(e^2-1)$, then $\sqrt{e^2-1} = \dfrac{b}{a}$. Hence the gradient of the tangent is $\dfrac{be}{a\left(\dfrac{b}{a}\right)} = e$.

(b) The tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$. Therefore the point A has coordinates $(0, b\,\mathrm{cosec}\,\theta)$. The normal to the



ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. Hence the point B has coordinates $\left(0, \frac{b^2-a^2}{b}\sin\theta\right)$.

(i) Gradient $AS \times$ gradient $BS = \frac{b\operatorname{cosec}\theta}{-ae}\cdot\frac{(b^2-a^2)\sin\theta}{b(-ae)}=\frac{(b^2-a^2)}{a^2e^2}$. Since for the ellipse $b^2=a^2(1-e^2)$, then gradient $AS \times$ gradient $BS=-1$. Hence AB subtends a right angle at S.

(ii) Since AB subtends a right angle at P, then A, P, S, B are concyclic with AB the diameter of the circle through the points. The centre of the circle is the midpoint of AB.

4 Solution

(a)

(i) Let $(a+ib)^2=-3+4i$, a,b - real. Then $(a^2-b^2)+(2ab)i=-3+4i$. Equating real and imaginary parts, $a^2-b^2=-3$ and $ab=2$. Hence $a^2-\frac{4}{a^2}=-3 \Rightarrow a^4+3a^2-4=0$, or $(a^2+4)(a^2-1)=0$. Since a is real, then $a=1\Rightarrow b=2$ or $a=-1\Rightarrow b=-2$.

(ii) In order to solve the equation $z^2-3z+(3-i)=0$ we need to calculate the discriminant $\Delta=9-4(3-i)=-3+4i$. We know (see p.**(i)**) that Δ has square roots $\pm(1+2i)$. The quadratic formula: $z^2-3z+(3-i)=0$ has solutions $z=\frac{3\pm(1+2i)}{2}$. Hence $z=1-i$ or $z=2+i$.

(b)

(i) Let $z=1+i\sqrt{3}$. Then $z=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\operatorname{cis}\frac{\pi}{3}$. Similarly $1-i\sqrt{3}=\bar{z}=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)=2\operatorname{cis}\left(-\frac{\pi}{3}\right)$.

(ii) Using De Moivre's theorem $z^{10}=2^{10}\left(\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}\right)$. Therefore

$$\left(1+i\sqrt{3}\right)^{10}+\left(1-i\sqrt{3}\right)^{10}=z^{10}+(\bar{z})^{10}=2\,\mathrm{Re}(z^{10})=2^{11}\cos\frac{10\pi}{3}=2048\cdot\left(-\frac{1}{2}\right)=-1024$$

(c) **(i)** and **(ii)** Let $y_1=x^3-x+2$ and $y_2=mx$. If y_2 is a tangent to y_1, then $y_1=y_2$ and $y_1'=y_2'$ at the point of contact.

$$y_1=y_2 \Rightarrow x^3-x+2=mx\,, \qquad (1)$$

$$y_1'=y_2' \Rightarrow 3x^2-1=m. \qquad (2)$$

$(1){:}(2) \Rightarrow \dfrac{x^3-x+2}{3x^2-1}=x \Rightarrow x^3-x+2=3x^3-x \Rightarrow x^3=1 \Rightarrow x=1$;

$x=1 \Rightarrow y_1=1^3-1+2 \Rightarrow y_1=2$;

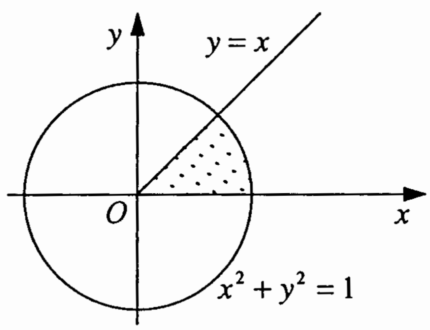
$x=1,\ y=2 \Rightarrow$ from $y_2=mx$ we have $m=2$.

So the equation of the tangent is $y=2x$, and the point of contact is (1,2).

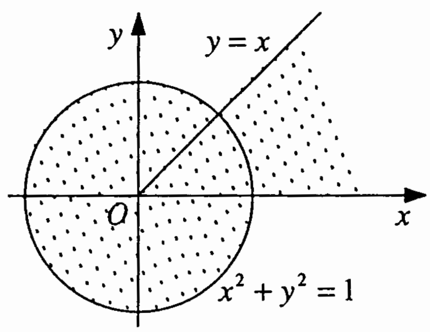
5 Solution

(a) $|z|=1$ is the circle, centre (0,0) and radius 1, $\arg z=0$ is the positive x-axis, $\arg z=\dfrac{\pi}{4}$ is the ray $y=x$, $x>0$.

(i)

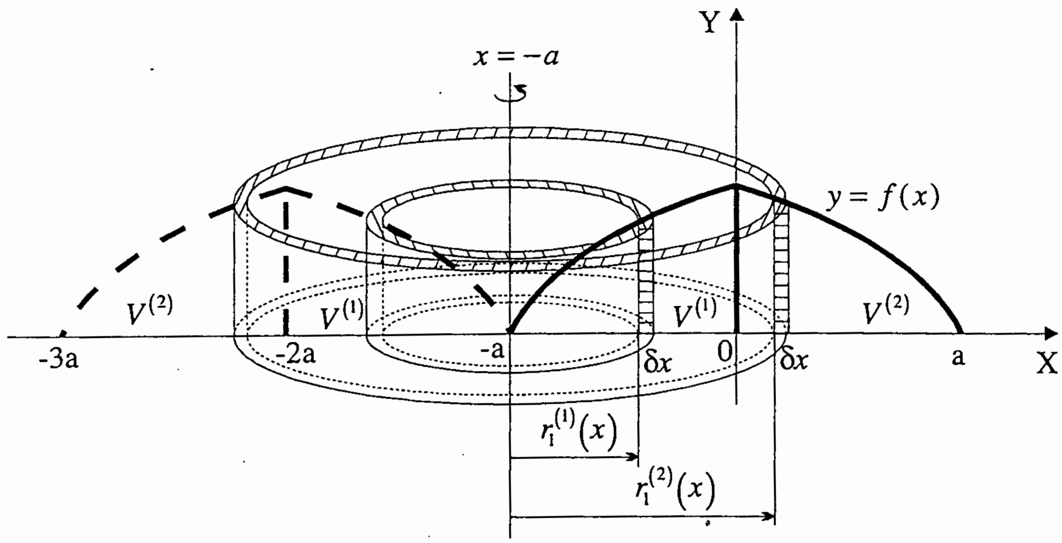


(ii)



(b) **(i)**

(b) (i)



Let us split the volume V of the solid into volumes $V^{(1)}$ and $V^{(2)}$ as shown on the figure. Now we shall find the volumes $V^{(1)}$ and $V^{(2)}$.

1) Volume $V^{(1)}$:

The typical cylindrical shell has radii $r_1^{(1)}(x)=x+a$, $r_2^{(1)}(x)=x+a+\delta x$, and height $h(x)=f(x)$. The shell has volume

$$\delta V^{(1)}=\pi\left[\left(r_2^{(1)}\right)^2-\left(r_1^{(1)}\right)^2\right]h(x)=2\pi(x+a)f(x)\delta x \quad \text{(ignoring } (\delta x)^2\text{)}.$$

$$\therefore \quad V^{(1)}=\lim_{\delta x\to 0}\sum_{x=-a}^{0}2\pi(x+a)f(x)\delta x=2\pi\int_{-a}^{0}(x+a)f(x)dx.$$

Substitution $x\to -x$ gives

$$V^{(1)}=-2\pi\int_{a}^{0}(-x+a)f(-x)dx.$$

Then $f(-x)=f(x) \;\Rightarrow\; V^{(1)}=2\pi\int_{0}^{a}(-x+a)f(x)dx$.

2) Volume $V^{(2)}$:

The typical cylindrical shell has radii $r_1^{(2)}(x)=x+2a$, $r_2^{(2)}(x)=x+2a+\delta x$, and height $h(x)=f(x)$. The shell has volume

$$\delta V^{(2)}=\pi\left[\left(r_2^{(2)}\right)^2-\left(r_1^{(2)}\right)^2\right]h(x)=2\pi(x+a)f(x)\delta x \quad \text{(ignoring } (\delta x)^2\text{)}.$$

$$\therefore \quad V^{(2)}=\lim_{\delta x\to 0}\sum_{x=0}^{a}2\pi(x+a)f(x)\delta x=2\pi\int_{0}^{a}(x+a)f(x)dx.$$

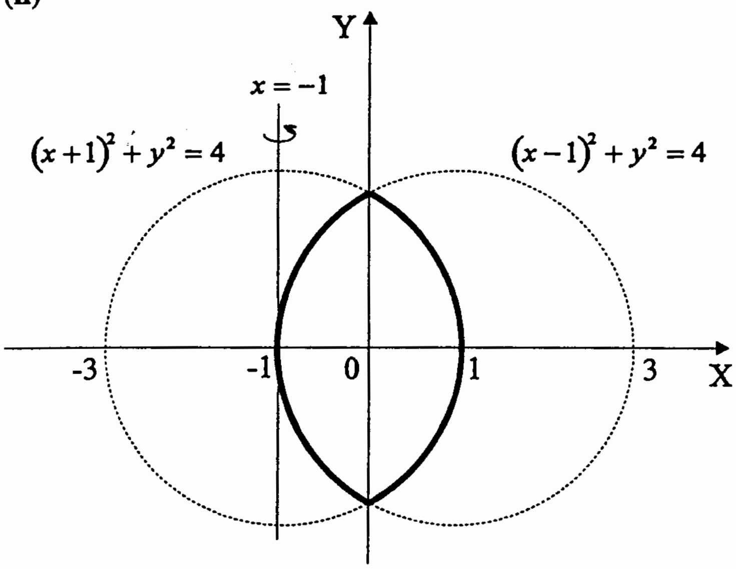
Thus

$$V=V^{(1)}+V^{(2)}=2\pi\int_{0}^{a}(-x+a)f(x)dx+2\pi\int_{0}^{a}(x+a)f(x)dx$$

$$= 2\pi\int_0^a [(-x+a)+(x+a)] f(x)\,dx = 4\pi a\int_0^a f(x)\,dx .$$

$\therefore$ the volume of such a solid is $4\pi a\int_0^a f(x)\,dx$.

(ii)



We can apply the formula derived above taking into account that the shape of the volume V to be found is symmetrical with respect to the plane $Y=0$. Hence

$$V = 2\cdot 4\pi a\int_0^a f(x)\,dx .$$

The equation of the left circle is

$$(x+1)^2+y^2=4 .$$

Therefore $f(x)=\sqrt{4-(x+1)^2}$, and $a=1$.

$$\therefore \quad V = 8\pi\int_0^1 \sqrt{4-(x+1)^2}\,dx .$$

We make the substitution $x=2\sin\varphi-1$, $dx=2\cos\varphi\,d\varphi$. Obtain new lower and upper limits of integration:

the lower limit: $x_l=0$

$0=2\sin\varphi_l-1$

$\varphi_l=\sin^{-1}(\tfrac{1}{2})=\dfrac{\pi}{6}$;

the upper limit: $x_u=1$

$1=2\sin\varphi_u-1$

$\varphi_u=\sin^{-1}(1)=\dfrac{\pi}{2}$.

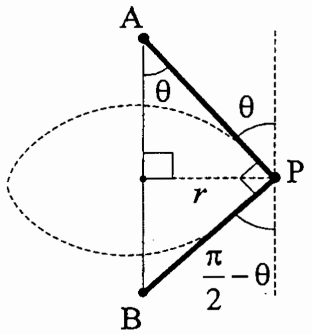
$$\therefore \quad V = 8\pi\cdot 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4-[(2\sin\varphi-1)+1]^2}\,\cos\varphi\,d\varphi = 32\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\varphi\,d\varphi = 32\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1+\cos 2\varphi}{2}\,d\varphi$$

$$=16\pi\left[\varphi+\frac{\sin 2\varphi}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = 16\pi\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(\frac{\pi}{6}+\frac{\sin(\frac{\pi}{3})}{2}\right)\right] = 16\pi\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right) = \frac{4\pi}{3}\left(4\pi-3\sqrt{3}\right)$$

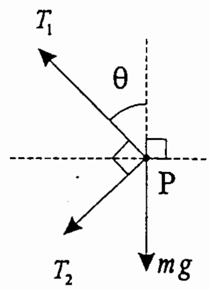
$\therefore$ the volume of the solid is $\dfrac{4\pi}{3}\left(4\pi-3\sqrt{3}\right)$ cubic cm.

6 Solution

Dimension diagram



Forces on P



(i) The resultant force on P is $mr\omega^2$ horizontally to the left. Its vertical component is

zero $\Rightarrow\ T_1\cos\theta - T_2\sin\theta = mg$. (1)

Its horizontal component is $mr\omega^2 \Rightarrow\ T_1\sin\theta + T_2\cos\theta = mr\omega^2$. (2)

$(1)\times\cos\theta + (2)\times\sin\theta \Rightarrow\ T_1 = m\omega^2 r\sin\theta + mg\cos\theta$. (3)

$(2)\times\cos\theta - (1)\times\sin\theta \Rightarrow\ T_2 = m\omega^2 r\cos\theta - mg\sin\theta$. (4)

Further, $AB = 5l$, $AP = 4l \Rightarrow\ BP = \sqrt{AB^2 - AP^2} = 3l$. Hence $\sin\theta = \dfrac{BP}{AB} = \dfrac{3}{5}$,

$\cos\theta = \dfrac{AP}{AB} = \dfrac{4}{5}$, and $r = AP\sin\theta = 4l\cdot\dfrac{3}{5} = \dfrac{12}{5}l$.

As the string is taut, $T_2 > 0 \Rightarrow$ from (4) $m\omega^2 r\cos\theta > mg\sin\theta \Rightarrow\ \omega^2\cdot\dfrac{12}{5}l\cdot\dfrac{4}{5} > g\cdot\dfrac{3}{5} \Rightarrow$

$16\omega^2 l > 5g$.

(ii) If the string is free to move, $T_1 = T_2$. Hence from (3) and (4)

$\Omega^2 r\sin\theta + g\cos\theta = \Omega^2 r\cos\theta - g\sin\theta \Rightarrow\ \Omega^2 r(\cos\theta - \sin\theta) = g(\cos\theta + \sin\theta) \Rightarrow$

$\Omega^2\dfrac{12}{5}l\left(\dfrac{4}{5} - \dfrac{3}{5}\right) = g\left(\dfrac{4}{5} + \dfrac{3}{5}\right) \Rightarrow 12\Omega^2 l = 35g$.

7 Solution

(a) (i) Let $R(x)$ be remainder, hence $\deg R < \deg(x-a)^2 = 2 \Rightarrow R(x) = cx + d$ with constants c and d. So $P(x) = (x-a)^2 Q(x) + cx + d$,

$P'(x) = (x-a)\{2Q(x) + (x-a)Q'(x)\} + c$.

From here
$$P(a) = ca + d\,, \qquad (1)$$
$$P'(a) = c\,. \qquad (2)$$

$(1) - (2) \times a \Rightarrow d = P(a) - a\,P'(a)$. Hence $cx + d = P'(a)x + P(a) - a\,P'(a)$.

(ii) $(x-1)$ is a factor of $P(x) \Rightarrow P(1) = 0 \Rightarrow 1 - 3 + k + 1 = 0 \Rightarrow k = 1$;

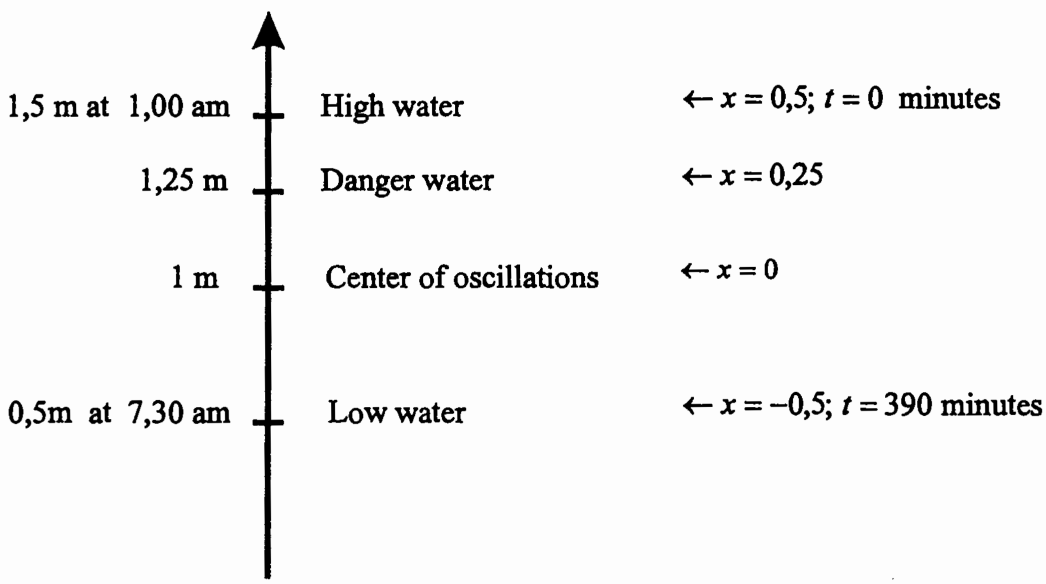
$P(x) = (x-1)Q(x)$ as $(x-1)$ is a factor of $P(x)$. Dividing $Q(x)$ by $(x-1)$, $Q(x) = (x-1)D(x) + c$, where c is constant. Hence

$$P(x) = (x-1)\{(x-1)D(x) + c\} \qquad (3)$$

$\Rightarrow P'(x) = \left\{(x-1)^2 D(x)\right\}' + c \Rightarrow P'(1) = c$. But $P'(x) = 11x^{10} - 18x^5 + 4x^3 + 2x \Rightarrow$

$P'(1) = 11 - 18 + 4 + 2 = -1$. Hence $c = -1$. So from (3) $P(x) = (x-1)^2 + 1 - x \Rightarrow$ the remainder is $(1-x)$.

(b)



Period T is $2 \cdot 390 = 780$ minutes.

Amplitude is $\frac{1}{2}(1{,}5 - 0{,}5) = 0{,}5\,\text{m}$.

Amplitude is $\frac{1}{2}(1{,}5-0{,}5)=0{,}5\,\text{m}$.

Motion is simple harmonic $\Rightarrow \ddot{x}=-n^2x,\ n=\frac{2\pi}{T}=\frac{\pi}{390}$.

This equation has solution $x=0{,}5\cos(nt+\alpha),\ 0\le\alpha<2\pi$.

Initial conditions: $t=0,\ x=0{,}5\Rightarrow\cos\alpha=1\Rightarrow\alpha=0\Rightarrow x=0{,}5\cos nt$.

There is a danger of flooding if $x\ge 0{,}25$. Find the first moment of time when

$x=0{,}25$; $0{,}25=0{,}5\cos nt\Rightarrow\cos nt=\frac{1}{2}\Rightarrow nt=\frac{\pi}{3}\Rightarrow t=130\,\text{minutes}=2{,}10$.

Hence $x=0{,}25$ at $1{,}00+2{,}10=3{,}10\,\text{am}$. So there is a danger of flooding from midnight to $3{,}10\,\text{am}$.

The following danger will be from $7{,}30+\left(\frac{T}{2}-2{,}10\right)=7{,}30+(6{,}30-2{,}10)=11{,}50\,\text{am}$

to $11{,}50+2\cdot 2{,}10=4{,}10\,\text{pm}$.

8 Solution

(a) If $u_1=5,\ u_2=13$ and $u_n=5u_{n-1}-6u_{n-2}$ for $n\ge 3$, show that $u_n=2^n+3^n$ for $n\ge 1$.

Define the statement $S(n)$: $\quad u_n=2^n+3^n$ for $n\ge 1$.

Consider $S(1)$: $n=1,\quad u_1=2+3=5\Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2,\quad u_2=2^2+3^2=13\Rightarrow S(2)$ is true.

Let k be a positive integer, $k\ge 2$. If $S(n)$ is true for all integer $n\le k$, then

$$u_n=2^n+3^n,\quad n=1,2,3,\dots,k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n=1,2,3,\dots,k$, we get

$$u_{k+1}=5u_k-6u_{k-1}=5(2^k+3^k)-6(2^{k-1}+3^{k-1})=2\cdot 2^k+3\cdot 3^k=$$
$$=2^{k+1}+3^{k+1}.$$

Hence for $k\ge 2$, $S(n)$ true for all positive integers $n\le k$ implies $S(k+1)$ is true. But $S(1)$, $S(2)$ are true. Therefore by induction, $S(n)$ is true for all positive integers n:

$$u_n=2^n+3^n\quad\text{for}\quad n\ge 1.$$

(b) **(i)** $\left(\sqrt{a}-\sqrt{b}\right)^2\ge 0\Rightarrow a+b-2\sqrt{ab}\ge 0\Rightarrow a+b\ge 2\sqrt{ab}$ (equality iff $a=b$).

(ii) multiplying these inequalities,

$$\left.\begin{aligned} a+b &\geq 2\sqrt{ab} \\ b+c &\geq 2\sqrt{bc} \\ c+a &\geq 2\sqrt{ca} \end{aligned}\right] \Rightarrow \quad (a+b)(b+c)(c+a) \geq 8abc \text{ (equality iff } a=b=c\text{)}.$$

Furthermore, $\frac{a}{b}+\frac{b}{c} \geq 2\sqrt{\frac{a}{c}}$, $\frac{c}{d}+\frac{d}{a} \geq 2\sqrt{\frac{c}{a}}$, $\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}} \geq 2$.

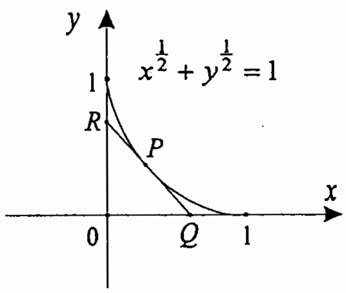
Hence, using these inequalities, we get

$$\left(\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{c}{d}+\frac{d}{a}\right) \geq 2\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right) \geq 4 \text{ (equality iff } a=b=c=d\text{)}.$$

Specimen Paper 2.

1 Solution

1 **(a)**



(i) Let us sketch the graph $x^{1/2}+y^{1/2}=1$.

It is clear that $x\ge 0$, $y\ge 0$. Hence

Domain $\{x\colon x\ge 0\}$, Range $\{y\colon y\ge 0\}$.

Features

- $y=0$ at $x=1$, $y=1$ at $x=0$
- $\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/2}$

As $x\to 0^+$, $y\to 1^-$, $\frac{dy}{dx}\to-\infty$.

The curve has a vertical tangent line at the critical point $(0,1)$.

As $x\to 1^-$, $y\to 0^+$, $\frac{dy}{dx}\to 0^-$.

The curve has a horizontal tangent line at the critical point $(1,0)$.

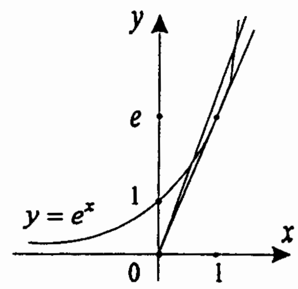
(ii) Let us show that $OQ+OR$ is independent of the position $P(x_1,y_1)$ (see Figure).

As $x^{1/2}+y^{1/2}=1$, we have $y=(1-\sqrt{x})^2=1-2\sqrt{x}+x$, $\frac{dy}{dx}=1-\frac{1}{\sqrt{x}}$.

The tangent at P is $y=(1-\frac{1}{\sqrt{x_1}})(x-x_1)+1-2\sqrt{x_1}+x_1=x(1-\frac{1}{\sqrt{x_1}})-\sqrt{x_1}+1$.

Hence $OQ+OR=\dfrac{\sqrt{x_1}-1}{1-\frac{1}{\sqrt{x_1}}}+1-\sqrt{x_1}=\dfrac{\sqrt{x_1}-1+1-\frac{1}{\sqrt{x_1}}-\sqrt{x_1}+1}{1-\frac{1}{\sqrt{x_1}}}=1$ ($OQ=x$ if $y=0$, and $OR=y$ if $x=0$), and it is independent of x_1.

(b)



(i) The tangent to the curve $y = f(x)$ which passes through the point $(x_0, y_0 = f(x_0))$ is given by $y = k(x-x_0)+y_0$, where $k = f'(x_0)$ is the gradient of the tangent.

If $y = f(x) = e^x$, $y_0 = e^{x_0}$, $k = e^{x_0}$, we get

$y = e^{x_0}(x-x_0)+e^{x_0}$.

As the tangent passes through the origin $(x=0, y=0)$, we obtain

$0 = -x_0e^{x_0}+e^{x_0} \Rightarrow x_0 = 1 \Rightarrow k = e^1 = e$.

Hence the tangent is $y = ex$.

(ii) If we take into account the solution described above, it is easily seen that the set of the values of the real number k for which the equation $e^x = kx$ has exactly two solutions, which are

$\{k\colon\ k > e\}$ (see Figure).

2 Solution

(a) $3+2x-x^2 = -(x^2-2x+1)+4 = 4-(x-1)^2 = 2^2-(x-1)^2 \Rightarrow b=2,\ a=1$.

Furthermore,

$I \equiv \int_1^3 \sqrt{(3+2x-x^2)}\,dx = \int_1^3 \sqrt{2^2-(x-1)^2}\,dx = 2\int_1^3 \sqrt{1-\left(\frac{x-1}{2}\right)^2}$, by using the substitution $u = \frac{x-1}{2}$, we get $I = 4\int_0^1 \sqrt{1-u^2}\,du$. Using the substitution $u = \sin t$ yields

$$I = 4\int_0^{\pi/2}\cos^2 t\,dt = 4\int_0^{\pi/2}\frac{1+\cos 2t}{2}\,dt = 2\left[t+\frac{\sin 2t}{2}\right]_0^{\pi/2} = \pi.$$

(b) We integrate by parts,

$$\int x\frac{1}{\cos^2 x}dx = \int x(\tan x)'\,dx = x\tan x - \int \tan x dx = x\tan x + \int\frac{d(\cos x)}{\cos x} =$$
$$x\tan x + \ln|\cos x|.$$

(c) We use the substitution $u = \tan\frac{x}{2}$;

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2},$$

$$1+\cos x = 1+2\cos^2\frac{x}{2}-1 = \frac{2}{1+\tan^2\frac{x}{2}} = \frac{2}{1+u^2};$$

$$u = \tan\frac{x}{2} \Rightarrow du = \frac{1}{2}\frac{dx}{\cos^2\frac{x}{2}} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow du = \frac{1}{2}\left(1+u^2\right)dx \Rightarrow dx = \frac{2du}{1+u^2}.$$

Hence

$$I \equiv \int_0^1 \frac{1}{1+\cos x+\sin x}dx = \int_0^1 \frac{1}{\frac{2}{1+u^2}+\frac{2u}{1+u^2}}\cdot\frac{2du}{1+u^2} = \int_0^1 \frac{du}{1+u} = \left[\ln|1+u|\right]_0^1 = \ln 2.$$

Furthermore, using the substitution $u = \frac{\pi}{2} - x$,

$$J \equiv \int_0^{\pi/2} \frac{x}{1+\cos x+\sin x}dx = -\int_{\pi/2}^0 \frac{\left(\frac{\pi}{2}-u\right)}{1+\cos\left(\frac{\pi}{2}-u\right)+\sin\left(\frac{\pi}{2}-u\right)}du = \int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-u\right)}{1+\sin u+\cos u}du =$$

$\frac{\pi}{2}\int_0^{\pi/2}\frac{1}{1+\cos u+\sin u}du - \int_0^{\pi/2}\frac{u}{1+\cos u+\sin u}du = \frac{\pi}{2}I - J$. So we have

$J = \frac{\pi}{2}I - J \Rightarrow J = \frac{\pi}{4}I$. But from the above $I = \ln 2$, hence $J = \frac{\pi}{4}\ln 2$.

3 Solution

(a)

(i) If $k < 4$, then $29 - k > 0$ and $4 - k > 0$. Therefore $\frac{x^2}{29-k} + \frac{y^2}{4-k} = 1$ is an ellipse with $a = \sqrt{29-k}$ and $b = \sqrt{4-k}$. Since $b < a$, then $b^2 = a^2(1-e^2)$. Hence $e = \frac{\sqrt{a^2-b^2}}{a}$ and the foci have coordinates $(\pm ae, 0) = \left(\pm\sqrt{a^2-b^2}, 0\right) = (\pm 5, 0)$. Thus the foci of the ellipse are independent of the value of k.

(ii) If $4 < k < 29$, then $29 - k > 0$ and $4 - k < 0$. Therefore $\frac{x^2}{29-k} + \frac{y^2}{4-k} = 1$ is a hyperbola with $a=\sqrt{29-k}$ and $b=\sqrt{k-4}$. For the hyperbola $b^2=a^2(e^2-1)$.

Hence $e=\dfrac{\sqrt{a^2+b^2}}{a}$ and the foci have coordinates $(\pm ae,0)=\left(\pm\sqrt{a^2+b^2},0\right)=(\pm 5,0)$. Thus the foci of the hyperbola are independent of the value of k.

(b) The tangent to the hyperbola $xy=c^2$ at the point $P\left(ct,\dfrac{c}{t}\right)$ has equation $x+t^2y=2ct$. Hence the tangent meets the x-axis at $Q(2ct,0)$ and the y-axis at $R\left(0,\dfrac{2c}{t}\right)$. The normal to the hyperbola $xy=c^2$ at the point $P\left(ct,\dfrac{c}{t}\right)$ has equation $tx-\dfrac{y}{t}=c\left(t^2-\dfrac{1}{t^2}\right)$. Thus the normal meets the line $y=x$ at $S\left(c\left(t+\dfrac{1}{t}\right),c\left(t+\dfrac{1}{t}\right)\right)$ and the line $y=-x$ at $T\left(c\left(t-\dfrac{1}{t}\right),-c\left(t-\dfrac{1}{t}\right)\right)$. Therefore

$$QS^2=c^2\left(t+\frac{1}{t}-2t\right)^2+c^2\left(t+\frac{1}{t}\right)^2=2c^2\left(t^2+\frac{1}{t^2}\right),$$

$$SR^2=c^2\left(t+\frac{1}{t}\right)^2+c^2\left(\frac{2}{t}-t-\frac{1}{t}\right)^2=2c^2\left(t^2+\frac{1}{t^2}\right),$$

$$RT^2=c^2\left(t-\frac{1}{t}\right)^2+c^2\left(-t+\frac{1}{t}-\frac{2}{t}\right)^2=2c^2\left(t^2+\frac{1}{t^2}\right),$$

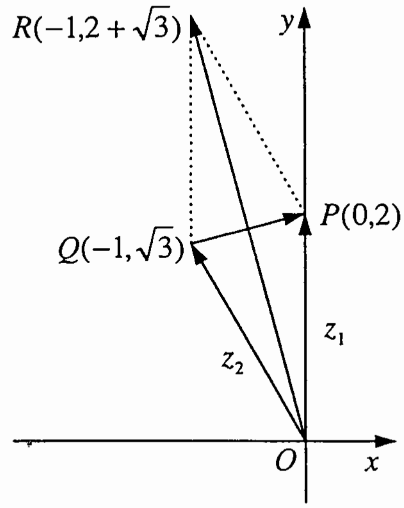
$$TQ^2=c^2\left(2t-t+\frac{1}{t}\right)^2+c^2\left(t-\frac{1}{t}\right)^2=2c^2\left(t^2+\frac{1}{t^2}\right).$$

So $QS=SR=RT=TQ$ and, consequently, $QSRT$ is a rhombus.

4 Solution

(a)

(i) The point P represents the number



$z_1 = 2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. So $z_1 = 2\,\text{cis}\frac{\pi}{2}$.

The point Q represents the number

$$z_2 = -1 + i\sqrt{3} = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

So $z_2 = 2\,\text{cis}\frac{2\pi}{3}$.

(ii) The vector $\overrightarrow{OR}$ represents the number $z_1 + z_2$.

The vector $\overrightarrow{QP}$ represents the number $z_1 - z_2$. Since $|z_1| = |z_2|$, then $OP = OQ = PR = QR$. Therefore $OPRQ$ is a rhombus. Hence $\angle POR = \frac{1}{2}\angle POQ$ and vector $\overrightarrow{QP}$ is obtained from vector $\overrightarrow{OR}$ by a rotation clockwise about O through $\frac{\pi}{2}$, followed by an enlargement in O by a certain factor. Thus

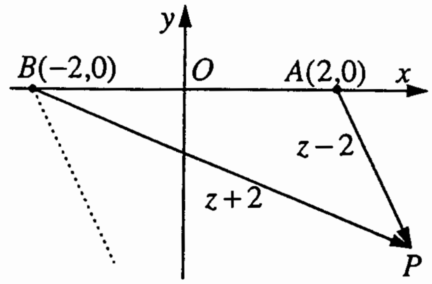
$$\arg(z_1 + z_2) = \frac{\pi}{2} + \angle POR = \frac{\pi}{2} + \frac{1}{2}\cdot\angle POQ.$$

Since $\angle POQ = \arg z_2 - \arg z_1 = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$, then $\arg(z_1 + z_2) = \frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{6} = \frac{7\pi}{12}$.

$$\arg(z_1 - z_2) = \arg(z_1 + z_2) - \frac{\pi}{2} = \frac{7\pi}{12} - \frac{\pi}{2} = \frac{\pi}{12}.$$

(b) Let $A(2,0)$, $B(-2,0)$ and P represent 2, -2 and z respectively. Then $\overrightarrow{AP}$ and $\overrightarrow{BP}$ represents $z-2$ and $z+2$ respectively. $\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$ requires $\overrightarrow{AP}$ to be parallel to the vector obtained by rotation of $\overrightarrow{BP}$ anticlockwise through an angle of $\frac{\pi}{4}$.

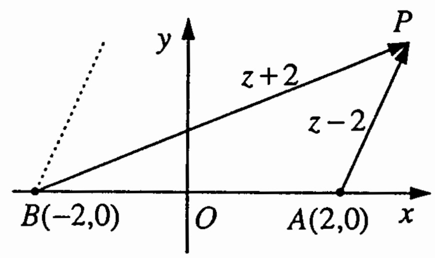
If P lies below the x-axis, AP must be parallel to a clockwise rotation of BP. This diagram shows



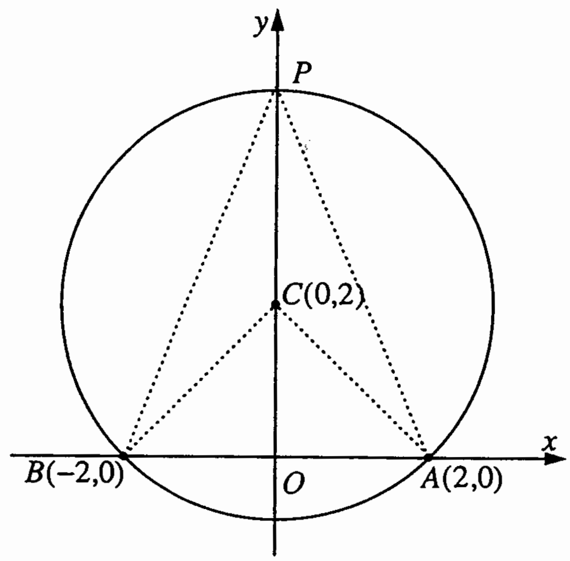
$\arg(z-2)-\arg(z+2)=-\frac{\pi}{4}$.

Hence P must lie above the x-axis.

Since alternate angles between parallel lines are equal, $\angle BPA=\frac{\pi}{4}$ as P traces its locus. Hence P lies on the major arc AB of a circle.



The centre C of this circle lies on the perpendicular bisector of AB, and chord AB subtends an angle $2\cdot\frac{\pi}{4}=\frac{\pi}{2}$ at C. Therefore $OC=2$ and $AC=2\sqrt{2}$. Hence the locus of P has equation $x^2+(y-2)^2=8$, $y>0$.



(c) Let the roots of $P(x)=x^3+ax^2-x-2$ be α, 2α and β. The production of the roots is

$$2\alpha^2\beta \Rightarrow 2\alpha^2\beta=2 \text{ and hence } \qquad \alpha^2\beta=1. \qquad (1)$$

The sum of products of the roots taken two at a time is $2\alpha^2+\alpha\beta+2\alpha\beta=2\alpha^2+3\alpha\beta$,

$$\text{hence } 2\alpha^2+3\alpha\beta=-1; \qquad (2)$$

$(2)\times\alpha \Rightarrow 2\alpha^3+3\alpha^2\beta=-\alpha$, and using (1), $2\alpha^3+3=-\alpha \Rightarrow 2\alpha^3+\alpha+3=0$. It is easy to see that $\alpha=-1$ is a zero of the last equation. To find other zeros, use the polynomial division

$2\alpha^3+\alpha+3$ by $\alpha+1$:

$$
\begin{array}{r}
2\alpha^2-2\alpha+3 \\
\alpha+1\overline{\big)\,2\alpha^3+\qquad\quad \alpha+3} \\
\underline{2\alpha^3+2\alpha^2\qquad\qquad} \\
-2\alpha^2+\alpha+3 \\
\underline{-2\alpha^2-2\alpha\quad} \\
3\alpha+3 \\
\underline{3\alpha+3} \\
0
\end{array}
$$

Hence $(2\alpha^3+\alpha+3)=(\alpha+1)(2\alpha^2-2\alpha+3)$. But $2\alpha^2-2\alpha+3=0 \Rightarrow \alpha=\dfrac{1\pm\sqrt{1-6}}{2}$.

And so the equation $2\alpha^3+\alpha+3=0$ has only one real zero $\alpha=-1$. Hence the second zero is $2\alpha=-2$ and from (1) the third zero is $\beta=\dfrac{1}{\alpha^2}=1$. Knowing all the roots of $P(x)$ we can factorise $P(x)$ over real numbers $P(x)=(x-\alpha)(x-2\alpha)(x-\beta)$, but $\alpha=-1,\ 2\alpha=-2,\ \beta=1$. Hence $P(x)=(x-1)(x+1)(x+2)$. Furthermore, the sum of roots $\alpha+2\alpha+\beta=3\alpha+\beta$. Hence $3\alpha+\beta=-a \Rightarrow a=-(-1\cdot 3+1)=2$.

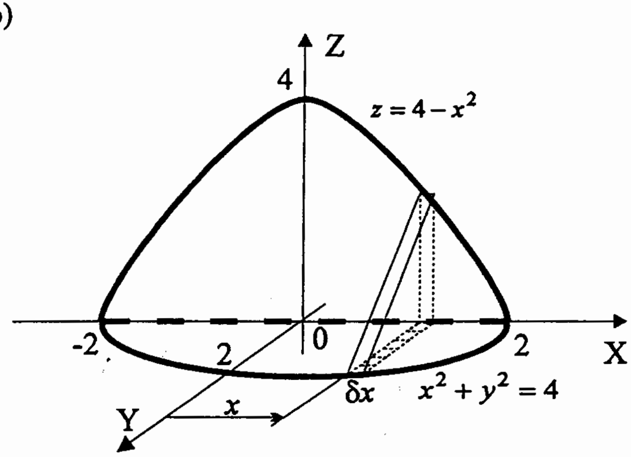
5 Solution

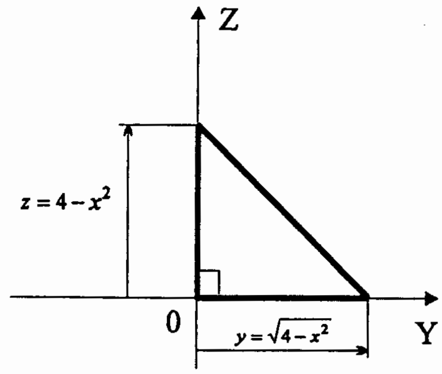
(a)

(i) Using De Moivre's theorem: $z^n=\cos n\theta+i\sin n\theta$ and $z^{-n}=\cos n\theta-i\sin n\theta$. Therefore $z^n+z^{-n}=2\cos n\theta$.

(ii) Since $2\cos\theta=z+z^{-1}$, then $16\cos^4\theta=\left(z+z^{-1}\right)^4=z^4+4z^2+6+4z^{-2}+z^{-4}$. But $z^4+z^{-4}=2\cos 4\theta$ and $z^2+z^{-2}=2\cos 2\theta$. Hence $16\cos^4\theta=2\cos 4\theta+8\cos 2\theta+6$. Thus $\cos^4\theta=\dfrac{1}{8}\left(\cos 4\theta+4\cos 2\theta+3\right)$.

(b)





The slice is a right-angled triangle with area of cross-section A, thickness δx.

$$A(x)=\frac{yz}{2}=\frac{\left(4-x^2\right)^{3/2}}{2}.$$

The slice has volume $\delta V = A(x)\delta x=\dfrac{\left(4-x^2\right)^{3/2}}{2}\delta x$. The volume of the solid is

$$V=\lim_{\delta x\to 0}\sum_{x=-2}^{2}\frac{\left(4-x^2\right)^{3/2}}{2}\delta x=\tfrac{1}{2}\int_{-2}^{2}\left(4-x^2\right)^{3/2}dx=\int_{0}^{2}\left(4-x^2\right)^{3/2}dx.$$

Substitution $x=2\sin\varphi$, $dx=2\cos\varphi\,d\varphi$ gives

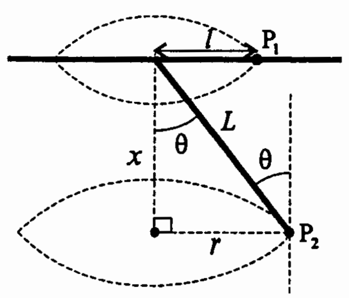
$$V=16\int_0^{\pi/2}\left(1-\sin^2\varphi\right)^{3/2}\cos\varphi\,d\varphi=16\int_0^{\pi/2}\cos^4\varphi\,d\varphi=16\int_0^{\pi/2}\left[\tfrac{1}{2}(1+\cos 2\varphi)\right]^2d\varphi$$

$$=4\int_0^{\pi/2}\left[1+2\cos 2\varphi+\tfrac{1}{2}(1+\cos 4\varphi)\right]d\varphi=4\left[\tfrac{3}{2}\varphi+\sin 2\varphi+\frac{\sin 4\varphi}{8}\right]_0^{\pi/2}=3\pi.$$

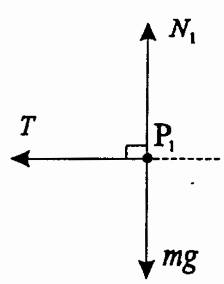
$\therefore$ the volume of the solid is 3π cubic units.

6 Solution

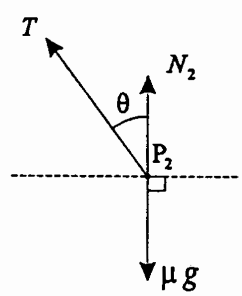
Dimension diagram



Forces on P_1



Forces on P_2



(a) The resultant force on P_1 is $m\omega^2 l$ horizontally to the left, hence $T = m\omega^2 l$. (1)

The resultant force on P_2 is $\mu\Omega^2 r$ horizontally to the left.

Its vertical component is zero, hence $T\cos\theta = \mu g - N_2$. (2)

Its horizontal component is $\mu\Omega^2 r$, hence $T\sin\theta = \mu\Omega^2 r$. (3)

(i) (3):(2) $\Rightarrow \tan\theta = \dfrac{\mu\Omega^2 r}{\mu g - N}$. But $\tan\theta = \dfrac{r}{a} \Rightarrow \dfrac{r}{x} = \dfrac{\mu\Omega^2 r}{\mu g - N} \Rightarrow N = \mu\left(g - x\Omega^2\right)$. It must be $N \geq 0 \Rightarrow g - x\Omega^2 \geq 0 \Rightarrow \Omega \geq \sqrt{\dfrac{g}{x}}$. Hence the maximum value of Ω is $\sqrt{\dfrac{g}{x}}$. If Ω reaches this maximum value, $N = 0$, and so the particle P_2 loses contact with floor.

(ii) (3):(1) $\Rightarrow \sin\theta = \dfrac{\mu\Omega^2 r}{m\omega^2 l}$. But $\sin\theta = \dfrac{r}{L} \Rightarrow \dfrac{l}{L} = \dfrac{\mu\Omega^2}{m\omega^2} \Rightarrow \dfrac{L}{l} = \dfrac{m}{\mu}\left(\dfrac{\omega}{\Omega}\right)^2$; $N = 0 \Rightarrow$ from (2) $T = \dfrac{\mu g}{\cos\theta}$. But $\cos\theta = \dfrac{x}{L} \Rightarrow T = \dfrac{\mu g L}{x}$.

(b) **(i)** From (a), (ii) $\dfrac{L}{l} = \dfrac{m}{\mu}\left(\dfrac{\omega}{\Omega}\right)^2$.

Hence $m = 0{,}4;\ \mu = 0{,}2;\ \omega = \Omega \Rightarrow \dfrac{L}{l} = \dfrac{0{,}4}{0{,}2} \Rightarrow L = 2l$.

But $L + l = 1{,}5 \Rightarrow 3l = 1{,}5 \Rightarrow l = 0{,}5 \Rightarrow L = 1$.

From (a), (ii) $T = \dfrac{\mu g L}{x}$. Hence $L = 1;\ x = 0{,}8;\ \mu = 0{,}2 \Rightarrow T = \dfrac{1}{4}g$.

(ii) From (3) $\Omega^2 = \dfrac{T\sin\theta}{\mu r}$. But $\sin\theta = \dfrac{r}{L}$, and $T = \dfrac{1}{4}g \Rightarrow \Omega^2 = \dfrac{g}{4\mu L}$. Hence $\mu = 0{,}2;\ L = 1 \Rightarrow \Omega^2 = \dfrac{5g}{4} \Rightarrow \Omega = \dfrac{\sqrt{5g}}{2}$. Hence the velocity v of the particle P_1 is

$$v = \Omega \cdot l \Rightarrow v = \frac{\sqrt{5g}}{2} \cdot 0{,}5 \Rightarrow v = \frac{1}{4}\sqrt{5g}.$$

7 Solution

(a) $P(x) = \left(x^2 - a^2\right) Q(x) + R(x)$, but $\deg R < \deg(x^2 - a^2) = 2 \Rightarrow R(x) = bx + c$ b, c are constants. So we have

$$P(x) = (x^2 - a^2)\,Q(x) + bx + c \;\Rightarrow P(a) = ba + c\,; \qquad (1)$$

$$\Rightarrow P(-a) = -ba + c \qquad (2)$$

$(1) + (2) \Rightarrow 2c = P(a) + P(-a) \Rightarrow c = \frac{1}{2}\{P(a) + P(-a)\}$;

$(1) — (2) \Rightarrow 2ba = P(a) - P(-a) \Rightarrow b = \frac{1}{2a}\{P(a) - P(-a)\}$.

Hence the remainder $R(x) = \frac{1}{2a}\{P(a) - P(-a)\} + \frac{1}{2}\{P(a) + P(-a)\}$.

(i) Let n be even and $P(x) = x^n - a^n$. Hence $P(a) = a^n - a^n = 0$ and as n even

$P(-a) = (-a)^n - a^n = 0$ So $(x - a)$ and $(x - (-a)) = (x + a)$ are factors of

$P(x) \Rightarrow (x - a)(x + a) = x^2 - a^2$ is a factor of $P(x) \Rightarrow$ the remainder is zero.

(ii) Let n be odd. Dividing P by $x^2 - a^2$,

$P(x) = (x - a)(x + a)Q(x) + bx + c;\ b, c$ are constants.

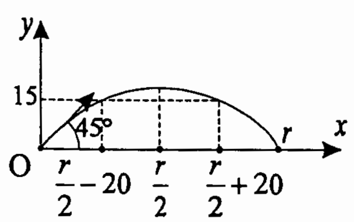
$$P(a) = a^n - a^n = 0 \Rightarrow ba + c = 0; \qquad (3)$$

$$P(-a) = (-a)^n - a^n = -2a^n \Rightarrow -ba + c = -2a^n; \qquad (4)$$

$(3) + (4) \Rightarrow 2c = -2a^n \Rightarrow c = -a^n$; $\qquad (3) - (4) \Rightarrow 2ba = 2a^n \Rightarrow b = a^{n-1}$.

Hence, the remainder is $bx + c = a^{n-1}x - a^n$.

(b) Axes, origin and trajectory $\qquad$ Equation of motion: $\ddot{x} = 0,\ \ddot{y} = -g$.



Initial conditions: when $t = 0$, $x = y = 0$;

$$\dot{x} = V \cdot \frac{1}{\sqrt{2}},\ \dot{y} = V \cdot \frac{1}{\sqrt{2}}.$$

After t seconds the particle is at position: $x = \frac{V}{\sqrt{2}}t$ $\quad$ (1), $\qquad y = \frac{V}{\sqrt{2}}t - \frac{gt^2}{2}$ $\quad$ (2)

From (1) $t = \frac{\sqrt{2}x}{V}$. Substituting this value of t into (2), $\qquad y = x - \frac{gx^2}{V^2}$; $\qquad$ (3)

$$x=\frac{r}{2}-20,\ y=15\Rightarrow \text{ from (3) } \left(\frac{r}{2}-20\right)-\frac{g}{V^2}\left(\frac{r}{2}-20\right)^2=15; \tag{4}$$

$$x=\frac{r}{2}+20,\ y=15\Rightarrow \text{ from (3) } \left(\frac{r}{2}+20\right)-\frac{g}{V^2}\left(\frac{r}{2}+20\right)^2=15; \tag{5}$$

$$(5)-(4)\Rightarrow\ 40-\frac{g}{V^2}\left\{\left(\frac{r}{2}+20\right)^2-\left(\frac{r}{2}-20\right)^2\right\}=0;$$

$$40V^2=g\left\{\left(\frac{r}{2}+20-\frac{r}{2}+20\right)\left(\frac{r}{2}+20+\frac{r}{2}-20\right)\right\}$$

$$40V^2=g(40r)$$

$$V^2=gr.$$

Hence, substituting $V^2=gr$ into (4),

$$\left(\frac{r}{2}-20\right)-\frac{1}{r}\left(\frac{r}{2}-20\right)^2=15;\ r\left(\frac{r}{2}-20\right)-\left(\frac{r^2}{4}-20r+400\right)=15r,\ \frac{r^2}{4}-15r-400=0,$$

$$r^2-60r-1600=0,\ r=30\pm\sqrt{900+1600},\ r=30\pm 50\Rightarrow r=80m.$$

8 Solution

(a) Show that 7^n+15^n is divisible by 11 for odd $n\geq 1$.

Solution

Let us introduce $f(n)=7^n+15^n$. It is easily seen that

$$f(n+2)=7^{n+2}+15^{n+2}=7^2(7^n+15^n)-49\cdot 15^n+225\cdot 15^n=49f(n)+176\cdot 15^n$$

For $n=1,3,5,\ldots$ let the statement $S(n)$ be defined by:
$f(n)$ is divisible by 11 for odd $n\geq 1$
Consider $S(1)$: $n=1$, $f(1)=7+15=22=11\cdot 2\Rightarrow S(1)$ is true, since $f(1)$ is divisible by 11.
Let k be a positive odd integer. If $S(k)$ is true for all integer k, then $f(k)=11\cdot M$ for some integer M. Consider $S(k+1)$. If $S(k)$ is true, we get

$$f(k+2)=49f(k)+15^k\cdot 176=49\cdot 11M+15^k\cdot 16\cdot 11=11(49\cdot M+15^k\cdot 16).$$

Since $49M+16\cdot 15^k$ is integer, we see that $f(k+2)$ is divisible by 11. Hence for all odd positive integers k, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Therefore by induction, $S(n)$ is true for all odd positive integers n: 7^n+15^n is divisible by 11 for odd $n\geq 1$.

(b) **(i)** $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0 \Rightarrow a+b-2\sqrt{ab} \geq 0 \Rightarrow a+b \geq 2\sqrt{ab}$ (equality iff $a=b$).

(ii) From (i) $a+b \geq 2\sqrt{ab} \Rightarrow ab \leq \dfrac{(a+b)^2}{4}$ and $2ab \leq \dfrac{(a+b)^2}{2}$.

Hence $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{b+a}{ab} \geq \dfrac{a+b}{\dfrac{(a+b)^2}{4}}=\dfrac{4}{a+b}=4$,

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{b^2+a^2}{(ab)^2}=\frac{(a+b)^2-2ab}{(ab)^2} \geq \frac{(a+b)^2-\dfrac{(a+b)^2}{2}}{\left\{\dfrac{(a+b)^2}{4}\right\}^2}=16\left(1-\frac{1}{2}\right)=8,$$

(equality iff $a=b=1/2$).

Specimen Paper 3.

1 Solution

(i) Let us sketch the graph of the function $y = \dfrac{3x}{x^2 - 1}$

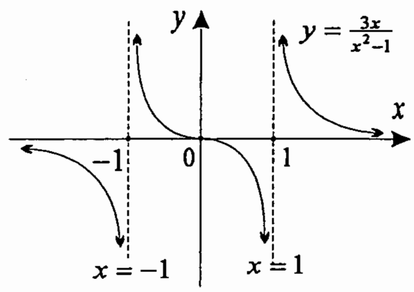
Features

- It is clear that $y(-x) = -y(x)$, that is, the function $y = \dfrac{3x}{x^2 - 1}$ is odd, and

$y = 0$ at $x = 0$. Its graph has point symmetry about the origin (it suffices to construct the graph for $x \geq 0$).

- $y \to -\infty$ as $x \to 1^-$, $y \to +\infty$ as $x \to 1^+$, $y \to 0^+$ as $x \to +\infty$
- Domain $\{x\colon\ x < -1,$ or $-1 < x < 1$ or $x > 1\}$

Range $\{y\colon -\infty < y < +\infty\}$



(ii) $\dfrac{3x}{x^2 - 1} = 2 \;\Rightarrow\; 3x - 2x^2 + 2 = 0 \;\Rightarrow\; x = \dfrac{3 \pm \sqrt{25}}{4} = -\dfrac{1}{2}, 2.$

Using the graph, let us solve the inequality $\dfrac{3x}{x^2 - 1} > 2$. It is easily seen that the solution is the set $\{x\colon -1 < x < -\dfrac{1}{2}$ or $1 < x < 2\}$.

(iii) If $y = \dfrac{3x}{x^2 - 1}$, then $k = y'(x_0) = -3\dfrac{x_0^2 + 1}{(x_0^2 - 1)^2}$ is the gradient of the curve. At the origin $x_0 = 0$, and $k = -3$. It is clear (see Figure) that the set of the values of the negative real number k for which the equation $\dfrac{3x}{x^2 - 1} = kx$ has exactly one real solution is $\{k\colon -3 < k < 0\}$.

(iv) The desired area S can be obtained as the integral

$$S = \int_0^{1/2} \frac{3x}{x^2-1} dx .$$

Using the substitution $1-x^2 = u$, we get

$$S = \frac{3}{2}\int_1^{3/4} \frac{du}{u} = \frac{3}{2}[\ln u]_1^{3/4} = \frac{3}{2}\left(\ln\frac{3}{4} - \ln 1\right) = \frac{3}{2}\ln\frac{3}{4} .$$

2 Solution

(a) Let us express $\frac{x+3}{(x+1)(x^2+1)}$ as a sum of partial fractions.

Let $\frac{x+3}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$, $\quad a,b,c$ constants.

Then $x+3 = a(x^2+1) + (bx+c)(x+1)$.

Putting $x=-1$ gives $a=1$.

Equate coefficients of x^2: $0 = a+b \Rightarrow b=-1$.

Equate constant terms: $3 = a+c \Rightarrow c=2$.

Hence $I \equiv \int \frac{x+3}{(x+1)(x^2+1)} dx = \int \frac{1}{x+1} dx + \int \frac{2-x}{x^2+1} dx$.

But $\int \frac{2-x}{x^2+1} dx = -\frac{1}{2}\int \frac{(x^2+1)'}{x^2+1} dx + 2\int \frac{dx}{x^2+1} = -\frac{1}{2}\ln|x^2+1| + 2\tan^{-1}x$, and

$\int \frac{dx}{x+1} = \ln|x+1|$.

Hence $I = \ln|x+1| - \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}x = \ln\left|\frac{x+1}{\sqrt{x^2+1}}\right| + 2\tan^{-1}x$.

(b) $x = 4\sin^2\theta \Rightarrow dx = 8\sin\theta\cos\theta d\theta$, $\theta = \sin^{-1}\frac{\sqrt{x}}{2} \Rightarrow \theta(0)=0$, $\theta(2) = \frac{\pi}{4}$. Hence

$$I \equiv \int_0^2 \sqrt{x(4-x)}dx = \int_0^{\pi/4} \sqrt{4\sin^2\theta\cdot(4-4\sin^2\theta)}\,8\sin\theta\cos\theta d\theta =$$

$$\int_0^{\pi/4} 2\sin\theta\cdot 2\cos\theta\cdot 8\sin\theta\cos\theta d\theta = 8\int_0^{\pi/4}\sin^2 2\theta d\theta =$$

$$8\int_0^{\pi/4}\frac{1-\cos 4\theta}{2}d\theta = 4\int_0^{\pi/4}d\theta - 4\int_0^{\pi/4}\cos 4\theta d\theta =$$

$$[4\theta]_0^{\pi/4} - \left[\frac{4\sin 4\theta}{4}\right]_0^{\pi/4} = \pi .$$

(c) **(i)** Use integration by parts:

$$I_n = \int_0^1 x^n e^x dx = \left[x^n e^x\right]_0^1 - \int_0^1 (x^n)' e^x dx = e - n\int_0^1 x^{n-1} e^x dx = e - n\, I_{n-1} .$$

(ii) $I_0 = \int_0^1 e^x dx = \left[e^x\right]_0^1 = e - 1$. From (i) $I_n = e - n\, I_{n-1}$.

Hence $n = 1 \Rightarrow I_1 = e - I_0 = e - (e-1) = 1$,

$n = 2 \Rightarrow I_2 = e - 2I_1 = e - 2\cdot 1 = e - 2$, $n = 3 \Rightarrow I_3 = e - 3I_2 = e - 3(e-2) = 6 - 2e$,

$n = 4 \Rightarrow I_4 = e - 4I_3 = e - 4(6-2e) = 9e - 24$.

Let $I \equiv \int_0^{1/2} x^4 e^{2x} dx$. Use the substitution $u = 2x \Rightarrow dx = \dfrac{du}{2} \Rightarrow$

$$I = \int_0^1 \left(\frac{u}{2}\right)^4 e^u \frac{du}{2} = \frac{1}{2^5}\int_0^1 u^4 e^4 du = \frac{1}{32} I_4 = \frac{9e - 24}{32} .$$

3 Solution

The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \dfrac{c}{t}\right)$ has equation

$$tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right).$$

(i) The point $Q\left(cq, \dfrac{c}{q}\right)$ lies on the normal. Hence $tcq - \dfrac{c}{tq} = c\left(t^2 - \dfrac{1}{t^2}\right)$. Therefore $\left(tq - t^2\right)\left(1 + \dfrac{1}{t^3 q}\right) = 0$. Since $Q \neq P$, then $q \neq t$. Thus $q = -\dfrac{1}{t^3}$ and Q has coordinates $\left(-\dfrac{c}{t^3}, -ct^3\right)$. Similarly the normal at Q cuts the hyperbola again at $R\left(cr, \dfrac{c}{r}\right)$ with $r = -\dfrac{1}{q^3} = t^9$. So R has coordinates $\left(ct^9, \dfrac{c}{t^9}\right)$.

(ii) The normal at P meets the x-axis at $A\left(\dfrac{c}{t}\left(t^2 - \dfrac{1}{t^2}\right), 0\right)$. The tangent to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. Hence the tangent meets the y-axis at $B\left(0, \frac{2c}{t}\right)$. If $M(x, y)$ is the midpoint of AB, then $x = \frac{c}{2t}\left(t^2 - \frac{1}{t^2}\right)$ and $y = \frac{c}{t}$. Thus $t = \frac{c}{y}$ and, consequently, $x = \frac{y}{2}\left(\frac{c^2}{y^2} - \frac{y^2}{c^2}\right)$. Therefore the locus of M has equation $2c^2 xy = c^4 - y^4$.

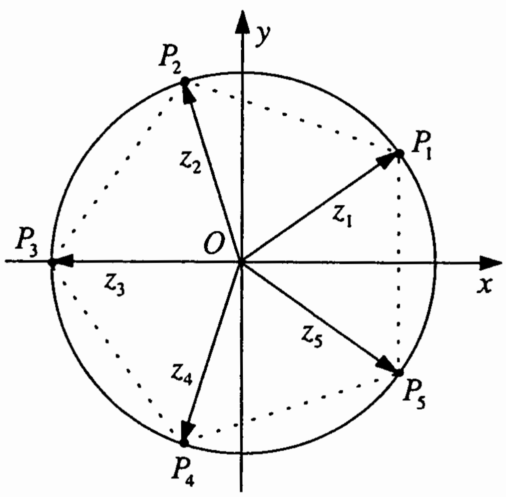
4 Solution

(a) Let $z = x + iy$, where x, y are real. Then $x^2 + y^2 + 2ix - 2y = 12 + 6i$. Equating real and imaginary parts, $x^2 + y^2 - 2y = 12$ and $2ix = 6i$. Hence $x = 3$ and $y^2 - 2y - 3 = 0$. Therefore the possible values of z are $3 - i$ and $3 + 3i$.

(b) $|-1| = 1$ and $\arg(-1) = \pi$. Hence the complex 5th roots of -1 all have modulus 1 and by De Moivre's theorem one complex 5th root of -1 has argument $\frac{\pi}{5}$, the others being equally spaced around the unit circle in the Argand diagram by an angle $\frac{2\pi}{5}$.

That is $z_1 = \text{cis}\frac{\pi}{5}$, $z_2 = \text{cis}\frac{3\pi}{5}$, $z_3 = -1$, $z_4 = \text{cis}\left(-\frac{3\pi}{5}\right)$, $z_5 = \text{cis}\left(-\frac{\pi}{5}\right)$.

(i) Let P_k represent the numbers z_k, $k = 1, \ldots, 5$. Since $OP_k = 1$ and $\angle P_k O P_{k+1} = \frac{2\pi}{5}$, $k = 1, \ldots, 5$, $(P_6 = P_1)$, then $P_k P_{k+1}$ is independent of k. Hence $P_1P_2P_3P_4P_5$ is a regular pentagon. Area of this pentagon is five times area of $\Delta P_1 O P_5$. That is area of the pentagon is $5 \cdot \frac{1}{2} \sin\frac{2\pi}{5}$.



(ii) We have well-known formula

$z^5 + 1 = (z + 1)(z^4 - z^3 + z^2 - z + 1)$. From the other hand

$z^5 + 1 = (z - z_3)(z - z_1)(z - z_5)(z - z_2)(z - z_4)$. Therefore

$$z^4 - z^3 + z^2 - z + 1 = (z - z_1)(z - z_5)(z - z_2)(z - z_4) = \left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$$

Since $z^4 - z^3 + z^2 - z + 1 = z^2\left[z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2}\right] = z^2\left[\left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) - 1\right]$, then

z_1, z_5, z_2, z_4 satisfy the equation $\left(z + \frac{1}{z}\right)^2 - \left(z + \frac{1}{z}\right) - 1 = 0$. But for the complex number on the unit circle $z + \frac{1}{z} = 2\operatorname{Re} z$. Therefore $\operatorname{Re} z_1 = \cos\frac{\pi}{5}$ and $\operatorname{Re} z_2 = \cos\frac{3\pi}{5}$ are the roots of the equation $4x^2 - 2x - 1 = 0$. Using quadratic formula, $x_1 = \frac{1}{4}\left(1 + \sqrt{5}\right)$ and $x_2 = \frac{1}{4}\left(1 - \sqrt{5}\right)$. Since $\cos\frac{\pi}{5} > 0$ and $\cos\frac{3\pi}{5} < 0$, then $\cos\frac{\pi}{5} = \frac{1}{4}\left(1 + \sqrt{5}\right)$ and $\cos\frac{3\pi}{5} = \frac{1}{4}\left(1 - \sqrt{5}\right)$.

5 Solution

(a) If $(x+1)^2$ is a factor of $P(x) = x^5 + 2x^2 + mx + n$, then $P(x) = (x+1)^2 Q(x)$. From here $P(-1) = 0$ and $P'(-1) = 0$ (-1 is a double root).

$P'(-1) = 0 \Rightarrow 5 - 4 + m = 0 \Rightarrow m = -1$.

$P(-1) = 0 \Rightarrow -1 + 2 - m + n = 0 \Rightarrow n = m - 1 \Rightarrow n = -2$.

(b) **(i)** If α, β and γ are roots of the equation $x^3 - px - q = 0$, then $\alpha + \beta + \gamma = 0$ and $\alpha\beta + \alpha\gamma + \beta\gamma = -p$. From here $0 = (\alpha + \beta + \gamma)^2 = (\alpha^2 + \beta^2 + \gamma^2) + 2(\alpha\beta + \alpha\gamma + \beta\gamma) \Rightarrow$ $\alpha^2 + \beta^2 + \gamma^2 = -2(\alpha\beta + \alpha\gamma + \beta\gamma) = 2p$.

(ii) $\alpha^3 - p\alpha - q = 0$, (Since α, β and γ are the roots of the given equation)

$\beta^3 - p\beta - q = 0$,

$\gamma^3 - p\gamma - q = 0$.

Summing these equalities, $(\alpha^3 + \beta^3 + \gamma^3) - p(\alpha + \beta + \gamma) - 3q = 0$.

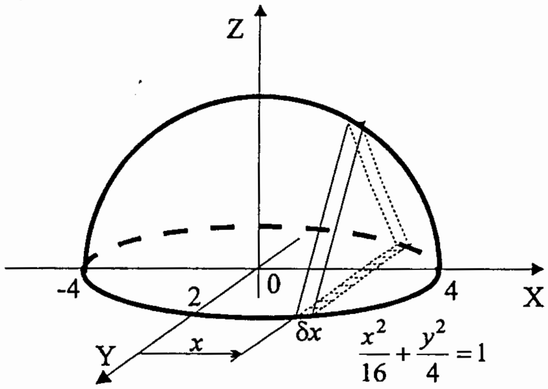
But from (i) $\alpha + \beta + \gamma = 0$, hence $\alpha^3 + \beta^3 + \gamma^3 = 3q$.

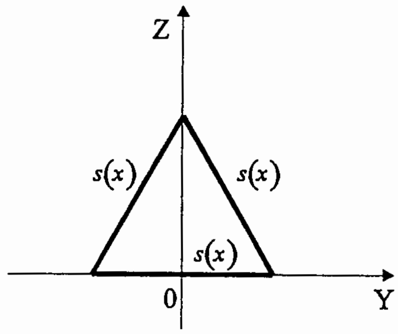
Furthermore, $\alpha^5 - p\alpha^3 - q\alpha^2 = 0$, (since the equation $x^2(x^3 - px - q) = 0$ has the roots $\beta^5 - p\beta^3 - q\beta^2 = 0$, $\alpha, \beta, \gamma, 0, 0$).

$$\gamma^5 - p\gamma^3 - q\gamma^2 = 0.$$

Summing these equalities, $\left(\alpha^5+\beta^5+\gamma^5\right) = p\left(\alpha^3+\beta^3+\gamma^3\right)+q\left(\alpha^2+\beta^2+\gamma^2\right)$. But $\alpha^2+\beta^2+\gamma^2 = 2p$ and $\alpha^3+\beta^3+\gamma^3 = 3q$. Hence $\alpha^5+\beta^5+\gamma^5 = p\cdot 3q + q\cdot 2p = 5pq$.

(c)





The slice is an equilateral triangle with area of cross-section A, thickness δx.

$$A(x) = \frac{\sqrt{3}\,s^2(x)}{4}$$

$$s(x) = 2\cdot 2\sqrt{1-\frac{x^2}{16}}$$

$$\therefore \quad A(x) = 4\sqrt{3}\left(1-\frac{x^2}{16}\right).$$

The slice has volume

$$\delta V = A(x)\,\delta x = 4\sqrt{3}\left(1-\frac{x^2}{16}\right)\delta x.$$

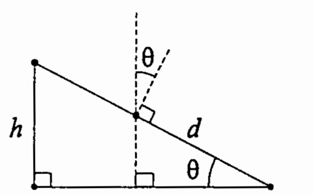
Then the volume of the solid is

$$V = \lim_{\delta x\to 0}\sum_{x=-4}^{4} 4\sqrt{3}\left(1-\frac{x^2}{16}\right)\delta x = 4\sqrt{3}\int_{-4}^{4}\left(1-\frac{x^2}{16}\right)dx = 4\sqrt{3}\left[x-\frac{x^3}{3\cdot 16}\right]_{-4}^{4} = \frac{64}{\sqrt{3}}.$$

$\therefore$ the volume of the solid is $\frac{64}{\sqrt{3}}$ cubic units.

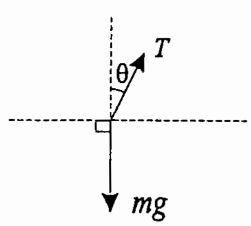
6 Solution

(a) Dimension diagram



$h = 0{,}1;\ d = 1{,}5;\ r = 500.$

Forces on train



Let v be an unknown velocity. The resultant force is $\frac{mv^2}{r}$ horizontally to the right.

Its vertical component is zero, hence $\quad T\cos\theta = mg$. (1)

Its horizontal component is $\frac{mv^2}{r}$, hence $T\sin\theta = \frac{mv^2}{r}$. (2)

$(2):(1) \Rightarrow \frac{v^2}{rg} = \tan\theta \Rightarrow v^2 = rg\tan\theta$. But $\tan\theta = \frac{h}{\sqrt{d^2-h^2}}$.

Hence $v^2 = rg\frac{h}{\sqrt{d^2-h^2}}$. Therefore,

$r = 500;\ h = 0{,}1;\ d = 1{,}5;\ g = 9{,}8 \Rightarrow v^2 = \frac{500\cdot 9{,}8\cdot 0{,}1}{\sqrt{1{,}5^2 - 0{,}1^2}} = 327{,}395 \Rightarrow v = 18{,}1\ ms^{-1}$.

(b) Choose the initial position as the origin and initial direction of motion as positive.

Equation of motion: $\ddot{x} = -k\left(1+v^2\right)$.

Initial conditions: $t = 0 \Rightarrow x = 0,\ v = u$.

(i) $\ddot{x} = -k\left(1+v^2\right) \Rightarrow \dot{v} = -k\left(1+v^2\right) \Rightarrow \frac{dv}{1+v^2} = -kdt \Rightarrow \tan^{-1} v = -kt + C,\ C$ is a constant.

$t = 0,\ v = u \Rightarrow C = \tan^{-1} u \Rightarrow t = \frac{1}{k}\left(\tan^{-1} u - \tan^{-1} v\right)$. As the particle is brought to rest, its velocity is zero. Hence $v = 0 \Rightarrow t = \frac{1}{k}\tan^{-1} u$.

(ii) $\ddot{x} = -k\left(1+v^2\right)$

$\Rightarrow v\frac{dv}{dx} = -k\left(1+v^2\right) \Rightarrow \frac{vdv}{1+v^2} = -kdx \Rightarrow \frac{1}{2}\int\frac{\left(1+v^2\right)'}{1+v^2}dv = -kx + A,\ A$ constant,

$\Rightarrow \frac{1}{2}\ln\left|1+v^2\right| = -kx + A$; $x = 0,\ v = u \Rightarrow A = \frac{1}{2}\ln\left|1+u^2\right|$; $\Rightarrow$

$x = \frac{1}{k}\left\{\frac{1}{2}\ln\left|1+u^2\right| - \frac{1}{2}\ln\left|1+v^2\right|\right\} \Rightarrow x = \frac{1}{2k}\ln\left(\frac{1+u^2}{1+v^2}\right)$; $v = 0 \Rightarrow x = \frac{1}{2k}\ln\left(1+u^2\right)$ is the travelled distance.

7 Solution

(a) **(i)** $\angle BAC = \angle BDC$ as these angles have a common arc BC. But also $\angle ABE = \angle DBC$. Hence the triangles ΔABE and ΔDBC have equal angles, and so these triangles are similar to each other. Furthermore, $\angle ADB = \angle ACB = \angle ECB$, as these angles have the common arc AB. Now

$\angle ABD - \angle EBC = (\angle ABE + \angle EBD) - (\angle EBD + \angle DBC) = \angle ABE - \angle DBC = 0 \Rightarrow$

$\angle ABD = \angle EBC$. Hence the angles of the triangles ΔADB and ΔBEC are equal, and so they are similar.

(ii)

Let ΔABE and ΔDBC be similar Let ΔADB and ΔEBC be similar

$$\Rightarrow \frac{AB}{DB} = \frac{EB}{BC} = \frac{AE}{DC}$$

$$\Rightarrow \frac{AD}{EC} = \frac{AB}{EB} = \frac{DB}{BC} \Rightarrow AB \cdot DC = DB \cdot AE \qquad (1)$$

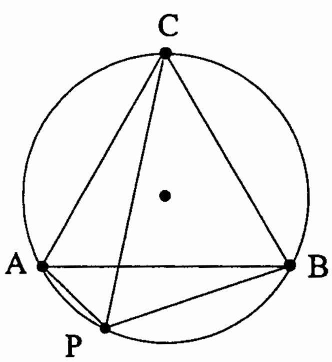
$$\Rightarrow AD \cdot BC = EC \cdot DB \qquad (2)$$

$(1) + (2)$

$\Rightarrow AB \cdot DC + AD \cdot BC = DB \cdot AE + EC \cdot DB \Rightarrow AB \cdot DC + AD \cdot BC = (AE + EC) \cdot DB$.

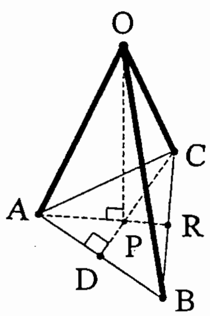
But $AE + EC = AC$. Hence $AB \cdot DC + AD \cdot BC = AC \cdot DB$.

(b)



Consider a cyclic quadrilateral APBC. From (ii) $AB \cdot PC = AC \cdot PB + BC \cdot PA$. But AB=AC=BC. Hence PC=PB+PA.

8 Solution



(a)

(i) $OA = OB \Rightarrow \angle OAB = \angle OBA \Rightarrow \angle OAB = \frac{1}{2}(180° - \angle AOB) = \frac{1}{2}(180° - 60°) = 60°$. Hence OAB is an equilateral triangle.

Analogously, OBC and OCA are equilateral. Hence AB=BC=CA=12. Let P be the projection of the point O onto the plane of the table, i.e. onto the plane of ΔABC. Hence $\angle OPA = \angle OPB = \angle OPC = 90°$. So the triangles OPA, OPB and OPC are right-angular. They also have the common side OP, and OA=OB=OC.

Hence $\Delta OPA = \Delta OPB = \Delta OPC \Rightarrow AP = BP = CP \Rightarrow$ CD and AR are medians of $\Delta ABC \Rightarrow PD = \frac{1}{3}CD$ and $AD = \frac{1}{2}AB = 6$. Furthermore, ΔADC is right-angular

$\Rightarrow CD^2 = AC^2 - AD^2 \Rightarrow CD^2 = 12^2 - 6^2 \Rightarrow CD = \sqrt{108}$;

$PD = \frac{1}{3}CD \Rightarrow PD = \sqrt{\frac{108}{9}} = \sqrt{12}$.

ΔADP is right-angular $\Rightarrow AP^2 = AD^2 + PD^2 \Rightarrow AP^2 = 6^2 + 12 \Rightarrow AP^2 = 48$.

ΔAPO is right-angular

$\Rightarrow OP^2 = OA^2 - AP^2 \Rightarrow OP^2 = 12^2 - 48 \Rightarrow OP = \sqrt{96} \Rightarrow OP = 4\sqrt{6}$.

(ii) $\tan\angle OAP = \frac{OP}{AP} = \frac{4\sqrt{6}}{\sqrt{48}} = \sqrt{2}$.

(b) **(i)** According to a well-known Laplace probability formula the desired probability of that the six scores obtained will be 1, 2, 3, 4, 5, 6 in some order is equal to

$P = \frac{6!}{6^6} = \frac{5}{324}$, since in the denominator we have the total number of the elementary outcomes that is 6^6, and the numerator represents the amount of successes being equal to $6!$.

(ii) It is clear that the product of members is even if and only if one of them is even. Let B be an event of that at least one of the numbers is even. Hence, if we again use the Laplace formula, the desired probability is given by

$P(A)=1-P(\overline{A})=1-\frac{3^6}{6^6}=1-\frac{1}{2^6}=\frac{63}{64}$ because the event $\overline{A}$ (opposite to A) means that all the numbers are old.

(iii) The probability of an event B having two scores consistently of exactly two 6's is equal to $\frac{1}{6^2}$. Let C be an event if the other four scores were all odd numbers, and $P(C)=\frac{3^4}{6^4}$. The events B and C are independent, hence $P(B\cdot C)=P(B)\cdot P(C)$. To calculate the desired probability P we should take into account the number of ways $\binom{6}{2}$ to choose two tests with realisation of a score of 6. Hence

$$P=P(B\cdot C)\cdot\binom{6}{2}=\frac{1}{6^2}\cdot\frac{3^4}{6^4}\cdot\binom{6}{2}=\frac{1}{36}\cdot\frac{1}{2^4}\cdot\frac{6!}{2!4!}=\frac{5}{192}.$$

(iv) Let D be an event if the first five throws exclude 6, event E implies that the first five throws include three odd numbers, and F be an event if the last test gave 6. Hence the desired probability $P(D\cdot E\cdot F)=P(D\cdot E)\cdot P(F)$ (DE and F are independent). Moreover $P(D\cdot E)=P(D)\cdot P_D(E)$, where $P_D(E)$ is a conditional probability of the event E after realisation of D. According to the Laplace formula we get $P(D)=\frac{5^5}{6^5}$ (1, 2, 3, 4, 5 only appear), $P(F)=\frac{1}{6}$. To calculate $P_D(E)$ let us employ the Bernoulli formula implying 5 tests. Probability of success in each test (odd number appears) is equal to $p=\frac{3}{5}$, $q=1-p=\frac{2}{5}$. Hence

$$P_D(E)=\binom{5}{3}\cdot p^3\cdot q^2=\binom{5}{3}\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^2,$$ and finally,

$$P(D\cdot E\cdot F)=\frac{5^5}{6^5}\cdot\frac{5!}{3!2!}\frac{3^3\cdot 2^3}{5^5}\cdot\frac{1}{6}=\frac{5}{6^3}=\frac{5}{216}.$$

Specimen Paper 4.

1 Solution

(a) **(i)** Let α be a repeated root of $P(x)=x^3-6x^2+9x+c$, c real. Hence $P(\alpha)=0$ and $P(\alpha)'=0$

$P(\alpha)'=0 \Rightarrow 3\alpha^2-12\alpha+9=0 \Rightarrow \alpha=\dfrac{6\pm 3}{3} \Rightarrow \alpha=1$ or $\alpha=3$;

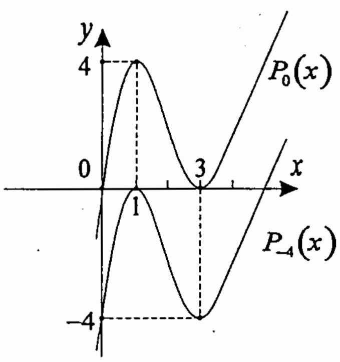
$\alpha=1 \Rightarrow P(1)=0 \Rightarrow 1-6+9+c=0 \Rightarrow c=-4$;

$\alpha=3 \Rightarrow P(3)=0 \Rightarrow 27-54+27+c=0 \Rightarrow c=0$.

(ii) Let $P_0(x)$ be $P(x)$ with $c=0$ and $P_{-4}(x)$ be $P(x)$ with $c=-4$.

$P_0(x)=0 \Rightarrow x=0$ or $x=3$; $P_0(x)'=0 \Rightarrow$ from (i) $x=1$ or $x=3$.

Hence $P_0(x)'=(x-1)(x-3)$. From here $P_0'(x)>0$ as $x\in(-\infty;1)\cup(3;+\infty) \Rightarrow$ here $P_0(x)\uparrow$, $P_0'(x)<0$ as $x\in(-1;3) \Rightarrow$ here $P_0(x)\downarrow$. Hence $x=1$ is a maximum point of $P_0(x)$, $P_0(1)=1-6+9=4$, and $x=3$ is a minimum point of $P_0(x)$, $P_0(3)=0$.



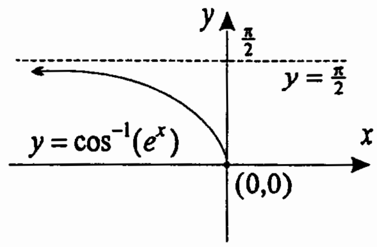
From the above we can sketch the graph of $P_0(x)$. Taking into account that $P_{-4}(x)=P_0(x)-4$,

we can also sketch the graph of $P_{-4}(x)$. $P(x)-P_0(x)=c \Rightarrow$ $P(x)>P_0(x)$ if $c>0$, and hence for $c>0$ the graph of $P(x)$ lies above the graph of $P_0(x)$.

$P(x)-P_{-4}(x)=c+4 \Rightarrow P(x)<P_{-4}(x)$ if $c<-4$, and hence for $c<-4$ the graph of $P(x)$ lies under the graph of $P_{-4}(x)$.

So, as seen from the graphs, the equation $P(x)=0$ has only one real root for $c\in(-\infty;-4)\cup(0;+\infty)$.

(b) **(i)** It is clear that the domain of the function $y=\cos^{-1}z$ is the set $\{z\colon -1\le z\le 1\}$, and the range is $\{y\colon 0\le y\le \pi\}$, besides $\cos^{-1}(-1)=\pi$, $\cos^{-1}(+1)=0$. Furthermore, the domain of the function $z=e^x$ is the set $\{x\colon -\infty<x<+\infty\}$, and the range is $\{z: 0 < z < +\infty\}$. Hence the domain of the function $y = \cos^{-1}(e^x)$ must be the set $\{x: -\infty < x \le 0\}$, since $z = e^x$ must satisfy the inequality $0 < z \le 1$ as the intersection of $\{z: -1 \le z \le 1\}$ and $\{z: 0 < z < +\infty\}$. As a result, the range of the function $y = \cos^{-1}(e^x)$ is the set $\{y: 0 \le y < \pi/2\}$, since $\cos^{-1}(e^0) = 0$, and $\cos^{-1}(e^x) \to \pi/2$ as $x \to -\infty$. .



(ii) Hence the graph of the function $y = \cos^{-1}(e^x)$ is represented by a monotonously decreasing curve.

2 Solution

(a) $x^2 + 2x + 5 = (x+1)^2 + 4 \Rightarrow a = 1, b = 2$;

$I \equiv \int_{-1}^{1} \frac{1}{x^2 + 2x + 5} dx = \int_{-1}^{1} \frac{1}{(x+1)^2 + 4} dx = \frac{1}{4} \int_{-1}^{1} \frac{1}{\left(\frac{x+1}{2}\right)^2 + 1} dx$. Use the substitution

$u = \frac{x+1}{2} \Rightarrow dx = 2du,\ u(-1) = 0,\ u(1) = 1$. Hence

$$I = \frac{1}{4} \int_0^1 \frac{1}{u^2 + 1} 2du = \left[\frac{1}{2} \tan^{-1} u\right]_0^1 = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}.$$

(b) Using integration by parts.

$$\int \frac{\ln x}{\sqrt{x}} dx = \int \ln x \left(2\sqrt{x}\right)' dx = 2\ln x \cdot \sqrt{x} - \int (\ln x)' \, 2\sqrt{x} dx = 2\sqrt{x} \ln x - \int \frac{2}{\sqrt{x}} dx = 2\sqrt{x} \ln x - 4\sqrt{x}.$$

(c) **(i)** Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{x(\pi - 2x)} dx$. Let us express $\frac{1}{x(\pi - 2x)}$ as a sum of partial fractions.

Let $\frac{1}{x(\pi - 2x)} = \frac{a}{x} + \frac{b}{\pi - 2x}$, a, b constants. Then $1 = a(\pi - 2x) + bx$.

Putting $x = 0$ gives $a = \frac{1}{\pi}$.

Putting $x = \frac{\pi}{2}$ gives $b = \frac{2}{\pi}$.

Hence $I = \frac{1}{\pi}\int_{\pi/6}^{\pi/3}\frac{1}{x}dx + \frac{2}{\pi}\int_{\pi/6}^{\pi/3}\frac{1}{\pi-2x}dx = \left[\frac{1}{\pi}\ln|x|\right]_{\pi/6}^{\pi/3} + \left[\frac{2}{\pi}\frac{\ln|\pi-2x|}{-2}\right]_{\pi/6}^{\pi/3} =$

$\frac{1}{\pi}\ln 2 - \frac{1}{\pi}\ln\frac{1}{2} = \frac{2\ln 2}{\pi}$.

(ii) Let $J = \int_{\pi/6}^{\pi/3}\frac{\cos^2 x}{x(\pi-2x)}dx$; $u = \frac{\pi}{2} - x \Rightarrow du = -dx, u\left(\frac{\pi}{6}\right) = \frac{\pi}{3}, u\left(\frac{\pi}{3}\right) = \frac{\pi}{6}$. Hence

$$J = -\int_{\pi/3}^{\pi/6}\frac{\cos^2\left(\frac{\pi}{2}-u\right)}{\left(\frac{\pi}{2}-u\right)2u}du = \int_{\pi/6}^{\pi/3}\frac{\sin^2 u}{(\pi-2u)u}du;$$

$$\Rightarrow 2J = \int_{\pi/6}^{\pi/3}\frac{\cos^2 x}{x(\pi-2x)}dx + \int_{\pi/6}^{\pi/3}\frac{\sin^2 u}{(\pi-2u)u}du =$$

$$= \int_{\pi/6}^{\pi/3}\frac{\cos^2 x + \sin^2 x}{x(\pi-2x)}dx = \int_{\pi/6}^{\pi/3}\frac{1}{(\pi-2x)x}dx = I \Rightarrow J = \frac{I}{2} = \frac{\ln 2}{\pi}, \text{ as } I = \frac{2\ln 2}{\pi} \text{ from (i).}$$

3 Solution

(a) Let $P(x_0, y_0)$ be the point of intersection. Then

P lies on the ellipse: $4x_0^2 + 9y_0^2 = 36$, (1)

P lies on the hyperbola: $4x_0^2 - y_0^2 = 4$. (2)

$(1)-(2) \Rightarrow 10y_0^2 = 32 \Rightarrow y_0^2 = 3.2$, (3)

$(1)+9\times(2) \Rightarrow 40x_0^2 = 72 \Rightarrow x_0^2 = 1.8$. (4)

Since $x_0^2 + y_0^2 = 5$, then the points of intersection of the ellipse and the hyperbola lie on the circle $x^2 + y^2 = 5$. The tangent to the ellipse at P has gradient $g_e = -\frac{4x_0}{9y_0}$ and the tangent to the hyperbola at P has gradient $g_h = \frac{4x_0}{y_0}$. Therefore, using (3),(4) we obtain $g_e \cdot g_h = -\frac{16x_0^2}{9y_0^2} = -\frac{16\cdot 1.8}{9\cdot 3.2} = -1$. Hence the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ meet at right angles.

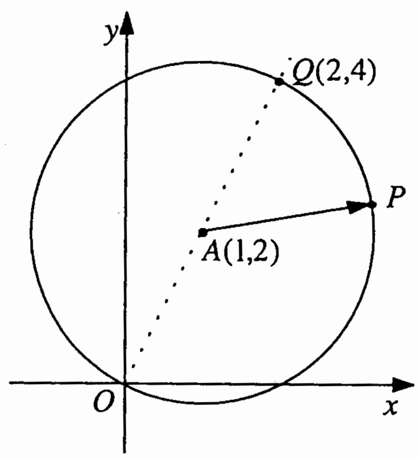
(b) The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. Let e be the eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the tangent to the hyperbola passes through the focus $S(\pm ae,0)$ of the ellipse. Then $\pm e\sec\theta=1$ and consequently $|\tan\theta|=\sqrt{\sec^2\theta-1}=\sqrt{\frac{1}{e^2}-1}=\frac{\sqrt{1-e^2}}{e}=\frac{b}{ae}$.

Hence the tangent to the hyperbola has equation $\pm\frac{x}{ae}-\frac{y}{ae}=1$ or $\pm\frac{x}{ae}+\frac{y}{ae}=1$. So the tangent is parallel to the line $y=x$ or to the line $y=-x$. Then the point $P(a\sec\theta,b\tan\theta)$ has coordinates $\left(\pm\frac{a}{e},\frac{b^2}{ae}\right)$ or $\left(\pm\frac{a}{e},-\frac{b^2}{ae}\right)$. Therefore the point P lies on the directrix $x=\frac{a}{e}$ or $x=-\frac{a}{e}$ of the ellipse.

4 Solution

(a) $z_1=\frac{7+4i}{3-2i}=\frac{(7+4i)(3+2i)}{(3-2i)(3+2i)}=\frac{(21-8)+(14+12)i}{9+4}=1+2i$.

Let A represent $z_1=1+2i$. Then $\vec{AP}$ represents $z-z_1$ and $|z-z_1|=\sqrt{5}\Rightarrow AP=\sqrt{5}$. Hence P lies on the circle centre $A(1,2)$ and radius $\sqrt{5}$. So the locus of P has equation $(x-1)^2+(y-2)^2=5$. Let Q be the intersection of the line OA and the circle. Then the greatest value of $|z|$ is $OQ=2\sqrt{5}$.



(b) Let $A(0,1),P$ represent i,z respectively. Then $|z-i|=AP$, and $\operatorname{Im} z$ is the distance from P to the line $y=0$. Hence the locus of P is a parabola, focus $A(0,1)$ and directrix $y=0$, with equation $x^2=2\left(y-\frac{1}{2}\right)\Rightarrow y=\frac{1}{2}\left(x^2+1\right)$. The tangent to the parabola at the point $P(x_0,y_0)$ has equation $xx_0=y+y_0-1$. Since the tangent passes through the origin, $y_0=1$. Thus the gradient of the tangent is $x_0=1$ or

$x_0 = -1$. Therefore the set of possible values of $\arg z$ is $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$.

(c) Let the desired polynomial be $P_3(x) = x^3 + d_2x^2 + d_1x + d_0$, d_0, d_1, d_2 constants.

$a+b+c=-3 \Rightarrow -d_2=-3 \Rightarrow d_2=3$; $abc=-6 \Rightarrow -d_0=-6 \Rightarrow d_0=6$;

$(a+b+c)=-3 \Rightarrow (a+b+c)^2=9 \Rightarrow (a^2+b^2+c^2)+2(ab+bc+ac)=9 \Rightarrow$

$ab+bc+ac=\frac{9-29}{2}=-10$; $ab+bc+ac=-10 \Rightarrow d_1=-10$.

Hence $P_3(x)=x^3+3x^2-10x+6$.

By inspection $P_3(1)=0 \Rightarrow (x-1)$ is a factor of $P_3(x)$. Use polynomial division

$$
\begin{array}{r}
x^2 + 4x - 6 \\
x-1 \overline{\smash{)}\, x^3 + 3x^2 - 10x + 6} \\
\underline{x^3 - x^2} \\
4x^2 - 10x + 6 \\
\underline{4x^2 - 4x} \\
-6x + 6 \\
\underline{-6x + 6} \\
0
\end{array}
$$

Hence $P_3(x)=(x-1)(x^2+4x-6)$; $P_3(x)=0 \Rightarrow x=1$ or $x^2+4x-6=0 \Rightarrow x=1$ or $x=-2\pm\sqrt{10}$. So the roots a,b,c are $1, -2\pm\sqrt{10}$.

5 Solution

(a) $z=(1+ic)^6=1+6ic-15c^2-20ic^3+15c^4+6ic^5-c^6$. If z is real then $\operatorname{Im} z=0$.

Hence $6c-20c^3+6c^5=0 \Rightarrow c(3c^4-10c^2+3)=0 \Rightarrow c(c^2-3)\left(c^2-\frac{1}{3}\right)=0$. So there are five real values of c for which z is real. They are 0, $\pm\sqrt{3}$, $\pm\frac{1}{\sqrt{3}}$.

(b) $P(x)=x^4+bx^3+cx^2+dx+e$, b,c,d,e real, is the polinomial with real coefficients. Hence

$z_1=(2+i)$ is a zero of $P(x) \Rightarrow z_2=\bar{z}_1=2-i$ and is a zero of $P(x)$ as well,

$z_3=(1-3i)$ is zero of $P(x) \Rightarrow z_4=\bar{z}_3=1+3i$ and is a zero of $P(x)$ as well.

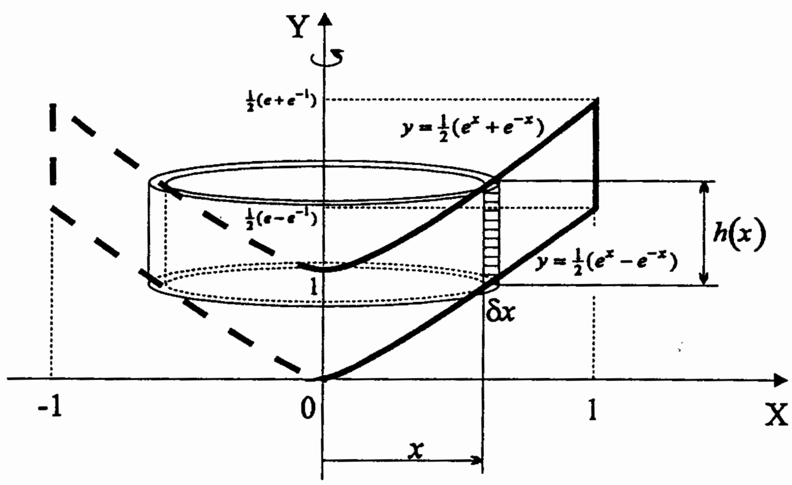
The product of the roots of $P(x)$ is

$$z_1 \cdot z_2 \cdot z_3 \cdot z_4 = (z_1\bar{z}_1)(z_3\bar{z}_3) = |z_1|^2|z_3|^2 = (4+1)(1+9) = 50 \Rightarrow e = 50.$$

The sum of the roots of $P(x)$ is

$$z_1 + z_2 + z_3 + z_4 = (z_1 + \bar{z}_1) + (z_3 + \bar{z}_3) = 2\operatorname{Re} z_1 + 2\operatorname{Re} z_3 = 2\cdot 2 + 2\cdot 1 = 6 \Rightarrow b = -6.$$

(c)



The typical cylindrical shell has radii x, $x+\delta x$, and height

$$h(x) = \tfrac{1}{2}\left(e^x + e^{-x}\right) - \tfrac{1}{2}\left(e^x - e^{-x}\right) = e^{-x}.$$

This shell has volume

$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]h(x) = 2\pi x\, e^{-x}\,\delta x \qquad \text{(ignoring } (\delta x)^2\text{)}.$$

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x\, e^{-x}\,\delta x = 2\pi \int_0^1 x\, e^{-x}\, dx = -2\pi \int_0^1 x\, de^{-x} = -2\pi\left[xe^{-x} + e^{-x}\right]_0^1$$

$$= 2\pi\left(1 - 2e^{-1}\right).$$

$\therefore$ the volume of the solid is $2\pi\left(1-2e^{-1}\right)$ cubic units.

6 Solution

(i) Origin is the point of projection. ↑ is the positive direction.

Initial conditions: $t=0,\ x=0,\ v=v_0$.

Equation of motion: $m\ddot{x} = -mg - mkv^2$.

$$\ddot{x} = -\left(g+kv^2\right) \Rightarrow \frac{vdv}{dx} = -\left(g+kv^2\right) \Rightarrow -dx = \frac{v}{g+kv^2}dv \Rightarrow -dx = \frac{1}{2k}\frac{\left(g+kv^2\right)'}{g+kv^2}dv \Rightarrow$$

$$-x+C=\frac{1}{2k}\ln\left(g+kv^2\right),\ C \text{ constant.}$$

$$x=0,\ v=v_0 \Rightarrow C=\frac{1}{2k}\ln\left(g+kv_0^2\right)\Rightarrow x=\frac{1}{2k}\ln\left(\frac{g+kv_0^2}{g+kv^2}\right).$$

Let h be the greatest height, then $x=h,\ v=0 \Rightarrow h=\frac{1}{2k}\ln\left(\frac{g+kv_0^2}{g}\right)$. (1)

(**ii**) The origin is the highest point. $\downarrow$ is the positive direction.

Initial conditions: $t=0,\ x=0,\ v=0$.

Equation of motion: $m\ddot{x}=mg-mkv^2$.

Terminal velocity: As $\ddot{x}\to 0,\ v\to\left(\frac{g}{k}\right)^{1/2}$, and hence $g-kv^2>0$.

$$\ddot{x}=\left(g-kv^2\right)\Rightarrow \frac{vdv}{dx}=\left(g-kv^2\right)\Rightarrow dx=\frac{v}{g-kv^2}dv \Rightarrow dx=\frac{-1}{2k}\frac{\left(g-kv^2\right)'}{g-kv^2}dv \Rightarrow$$

$$x+C=\frac{-1}{2k}\ln\left(g-kv^2\right),\ C \text{ constant.}$$

$$x=0,\ v=0 \Rightarrow C=-\frac{1}{2k}\ln g \Rightarrow x=\frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right). \quad (2)$$

As the particle reaches the ground, $x=h$.

$x=h$, $v=v_1 \Rightarrow$ from (2) $h=\frac{1}{2k}\ln\left(\frac{g}{g-kv_1{}^2}\right)\Rightarrow$ from (1)

$$\frac{1}{2k}\ln\left(\frac{g+kv_0^2}{g}\right)=\frac{1}{2k}\ln\left(\frac{g}{g-kv_1{}^2}\right)$$

$$\Rightarrow \left(g+kv_0^2\right)\left(g-kv_1^2\right)=g^2.$$

7 Solution

(**a**) (**i**) $\operatorname{cosec}2\theta=\frac{1}{\sin 2\theta}=\frac{\sin^2\theta+\cos^2\theta}{2\sin\theta\cos\theta}=\frac{\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}}{\frac{2\sin\theta\cos\theta}{\sin^2\theta}}=\frac{1+\cot^2\theta}{2\cot\theta}$,

$$\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta} = \frac{\dfrac{\cos^2\theta - \sin^2\theta}{\sin^2\theta}}{\dfrac{2\sin\theta\cos\theta}{\sin^2\theta}} = \frac{\cot^2\theta - 1}{2\cot\theta}.$$

Hence $\operatorname{cosec}2\theta + \cot 2\theta = \dfrac{1+\cot^2\theta}{2\cot\theta} + \dfrac{\cot^2\theta - 1}{2\cot\theta} \Rightarrow \operatorname{cosec}2\theta + \cot 2\theta = \cot\theta$.

(1)

(ii) Using (1),

$\cot\dfrac{\pi}{8} = \operatorname{cosec}\left(2\cdot\dfrac{\pi}{8}\right) + \cot\left(2\cdot\dfrac{\pi}{8}\right) \Rightarrow \cot\dfrac{\pi}{8} = \operatorname{cosec}\dfrac{\pi}{4} + \cot\dfrac{\pi}{4} = \sqrt{2}+1$.

Analogously, $\cot\dfrac{\pi}{12} = \operatorname{cosec}\dfrac{\pi}{6} + \cot\dfrac{\pi}{6} \Rightarrow \cot\dfrac{\pi}{12} = 2+\sqrt{3}$. From (1)

$\operatorname{cosec}2\theta = \cot\theta - \cot 2\theta$. Hence $\operatorname{cosec}\dfrac{2\pi}{15} = \cot\dfrac{\pi}{15} - \cot\dfrac{2\pi}{15}$,

$\operatorname{cosec}\dfrac{4\pi}{15} = \cot\dfrac{2\pi}{15} - \cot\dfrac{4\pi}{15}$, $\operatorname{cosec}\dfrac{8\pi}{15} = \cot\dfrac{4\pi}{15} - \cot\dfrac{8\pi}{15}$,

$\operatorname{cosec}\dfrac{16\pi}{15} = \cot\dfrac{8\pi}{15} - \cot\dfrac{16\pi}{15}$. Adding together these four equalities,

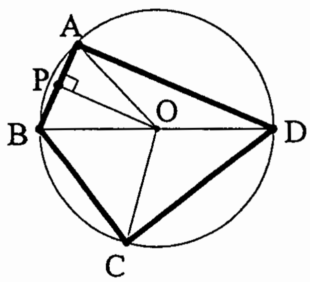
$\operatorname{cosec}\dfrac{2\pi}{15} + \operatorname{cosec}\dfrac{4\pi}{15} + \operatorname{cosec}\dfrac{8\pi}{15} + \operatorname{cosec}\dfrac{16\pi}{15} = \cot\dfrac{\pi}{15} - \cot\dfrac{16\pi}{15} = \cot\dfrac{\pi}{15} - \cot\left(\pi + \dfrac{\pi}{15}\right) =$

$\cot\dfrac{\pi}{15} - \cot\dfrac{\pi}{15} = 0.$

(b)

(i) $\angle AOB = \alpha$, $\angle BOC = \alpha+\beta$, $\angle COD = \alpha + 2\beta$, $\angle DOA = \alpha + 3\beta \Rightarrow$

$\angle AOB + \angle BOC + \angle COD + \angle DOA = 4\alpha + 6\beta$.



From the other hand

$\angle AOB + \angle BOC + \angle COD + \angle DOA = 2\pi$.

Hence $4\alpha + 6\beta = 2\pi \Rightarrow 2\alpha + 3\beta = \pi$. (2)

But

$\angle AOB + \angle DOA = 2\alpha + 3\beta \Rightarrow \angle AOB + \angle DOA = \pi \Rightarrow$

the points B, O and D lie on a straight line.

(ii) Let $S_{\Delta OAB}$ be the area of the triangle OAB. If $\angle APO = 90^\circ$, then

$S_{\Delta OAB} = \dfrac{1}{2}AB\cdot OP$, but $AB = 2AP = 2r\sin\dfrac{\alpha}{2}$ and $OP = r\cos\dfrac{\alpha}{2}$. Hence

$S_{\Delta OAB} = \frac{1}{2}\left(2r\sin\frac{\alpha}{2}\right)r\cos\frac{\alpha}{2} \Rightarrow S_{\Delta OAB} = \frac{r^2}{2}\sin\alpha$. Analogously, $S_{\Delta OBC} = \frac{r^2}{2}\sin(\alpha+\beta)$,

$S_{\Delta OCD} = \frac{r^2}{2}\sin(\alpha+2\beta)$ and $S_{\Delta ODA} = \frac{r^2}{2}\sin(\alpha+3\beta)$. As

$S_{ABCD} = S_{\Delta OAB} + S_{\Delta OBC} + S_{\Delta OCD} + S_{\Delta ODA}$, we have

$S_{ABCD} = \frac{r^2}{2}\left(\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \sin(\alpha+3\beta)\right)$. From (2) $\alpha = \frac{\pi}{2} - \frac{3\beta}{2} \Rightarrow$

$$S_{ABCD} = \frac{r^2}{2}\left(\sin\left(\frac{\pi}{2}-\frac{3\beta}{2}\right) + \sin\left(\frac{\pi}{2}-\frac{3\beta}{2}+\beta\right) + \sin\left(\frac{\pi}{2}-\frac{3\beta}{2}+2\beta\right) + \sin\left(\frac{\pi}{2}-\frac{3\beta}{2}+3\beta\right)\right) \Rightarrow$$

$$S_{ABCD} = \frac{r^2}{2}\left(\cos\frac{3\beta}{2} + \cos\frac{\beta}{2} + \cos\frac{\beta}{2} + \cos\frac{3\beta}{2}\right) \Rightarrow S_{ABCD} = r^2\left(\cos\frac{3\beta}{2} + \cos\frac{\beta}{2}\right) =$$

$$r^2\left(\cos\left(\beta+\frac{\beta}{2}\right) + \cos\frac{\beta}{2}\right) = r^2\left(\cos\beta\cos\frac{\beta}{2} - \sin\beta\sin\frac{\beta}{2} + \cos\frac{\beta}{2}\right).$$

But $\cos\frac{\beta}{2} - \sin\frac{\beta}{2}\sin\beta = \cos\frac{\beta}{2} - \sin\frac{\beta}{2}\cdot 2\sin\frac{\beta}{2}\cos\frac{\beta}{2} = \cos\frac{\beta}{2}\left(1-2\sin^2\frac{\beta}{2}\right) = \cos\frac{\beta}{2}\cos\beta$.

Hence $S_{ABCD} = 2r^2\cos\beta\cos\frac{\beta}{2}$.

8 Solution

(a) If $u_1 = 1$ and $u_n = \sqrt{3u_{n-1}}$ for $n \geq 2$,

(i) show that $u_n < 3$ for $n \geq 1$.

For $n = 1,2,3,\ldots$ let the statement $S(n)$ be defined by: $u_n < 3$ for $n \geq 1$.
Consider $S(1)$: $n=1$, $u_1 = 1 < 3 \Rightarrow S(1)$ is true. Let k be a positive integer. If $S(k)$ is true for all integer k, then $u_k < 2$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = \sqrt{3u_k} = \sqrt{3}\sqrt{u_k} < \sqrt{3}\sqrt{3} = 3,$$

because of $\sqrt{u_k} < \sqrt{3}$, and $u_{k+1} < 3$. Hence for all positive integers k, $S(k)$ true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers n: $u_n < 3$ for $n \geq 1$.

(ii) Deduce that $u_{n+1} > u_n$ for $n \geq 1$.

For $n = 1,2,3,\ldots$ let the statement $S(n)$ be defined by: $u_n < u_{n+1}$ for $n \geq 1$.
Consider $S(1)$: $n=1$, $u_1 < u_2$, since $u_1 = 1, u_2 = \sqrt{3}$. Hence $S(1)$ is true. Let k be a positive integer. If $S(k)$ is true for all integer k, then $u_k < u_{k+1}$ for $k \geq 1$. Consider $S(k+1)$. If $S(k)$ is true, we get

$$u_{k+1} = \sqrt{3u_k} < \sqrt{3u_{k+1}} = u_{k+2},$$

because of $\sqrt{u_k} < \sqrt{u_{k+1}}$, and $u_{k+1} < u_{k+2}$. Hence for all positive integers k, $S(k)$ is true implies $S(k+1)$ is true. But $S(1)$ is true. Hence by induction, $S(n)$ is true for all positive integers n: $u_n < u_{n+1}$ for $n \ge 1$.

(b) Notice that $(a-b)^2 \ge 0 \Rightarrow a^2+b^2-2ab \ge 0 \Rightarrow ab \le \dfrac{a^2+b^2}{2}$ and analogously

$$bc \le \frac{b^2+c^2}{2}, \qquad ca \le \frac{c^2+a^2}{2}. \qquad (1)$$

Summing these inequalities,

we get $ab+bc+ca \le a^2+b^2+c^2$ (equality iff $a=b=c$). (2)

Exchanging in (2) a by a^2, b by b^2 and c by c^2 yields

$a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2$.

Using (1), we obtain

$$abc(a+b+c) = a^2bc+b^2ac+c^2ab \le a^2\left(\frac{b^2+c^2}{2}\right)+b^2\left(\frac{a^2+c^2}{2}\right)+c^2\left(\frac{a^2+b^2}{2}\right)=$$

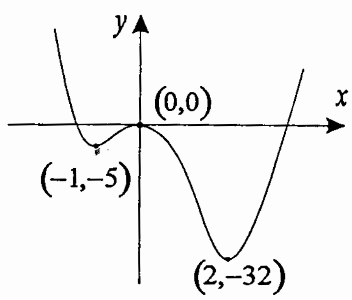
$$a^2b^2+b^2c^2+a^2c^2 \text{ (equality iff } a=b=c).$$

Specimen Paper 5.

1 Solution

(a) Let us sketch the graph of the function $y = 3x^4 - 4x^3 - 12x^2$. It is easily seen that $y \to +\infty$ as $x \to -\infty$, and $y \to +\infty$ as $x \to +\infty$. Find the roots of the equation $0 = 3x^4 - 4x^3 - 12x^2$. As $0 = x^2(3x^2 - 4x - 12)$, the roots are

$x = 0,\ x = (2 \pm 2\sqrt{10})/3$.



Find the turning points:

$y'(x) = 12x^3 - 12x^2 - 24x = 0 \Rightarrow$

$x^3 - x^2 - 2x = 0 \Rightarrow x = -1,\ x = 0,\ x = 2$

$y(-1) = -5$ (maximum),
$y(0) = 0$ (minimum),
$y(2) = -32$ (maximum).

(i) If k belongs to the set $\{k: -32 > k\}$, the equation $3x^4 - 4x^3 - 12x^2 - k = 0$ has no real roots.

(ii) If k belongs to the set $\{k: -32 < k < -5 \text{ or } k > 0\}$, the equation $3x^4 - 4x^3 - 12x^2 - k = 0$ has just two distinct real roots.

(iii) If k belongs to the set $\{k: -5 < k < 0\}$, the equation $3x^4 - 4x^3 - 12x^2 - k = 0$ has four distinct real roots.

(b) **(i)** Let $P(x) = 3x^4 - 4x^3 - 12x^2 - k$. Then

$P(1) = 0 \Rightarrow 3 - 4 - 12 - k = 0 \Rightarrow k = -13$.

Hence $(x-1)$ is a factor of $P(x) = 3x^4 - 4x^3 - 12x^2 + 13$.

Using the following polynomial division,

$$
\begin{array}{r}
3x^3 - x^2 - 13x - 13 \\
x-1\overline{)\,3x^4 - 4x^3 - 12x^2 + 13} \\
\underline{3x^4 - 3x^3 \qquad\qquad\qquad} \\
-x^3 - 12x^2 + \;\; 13 \\
\underline{-x^3 + \quad x^2 \qquad\quad} \\
-13x^2 + \;\; 13 \\
\underline{-13x^2 + 13x} \\
-13x + 13 \\
\underline{-13x + 13} \\
0
\end{array}
$$

So $P(x) = (x-1)Q(x)$, where

$Q(x) = 3x^3 - x^2 - 13x - 13$.

$$Q(0) < 0,\ Q(3) > 0 \Rightarrow 0 < \alpha < 3;$$

$$Q(1) < 0,\ Q(2) < 0 \Rightarrow 2 < \alpha < 3.$$

(ii) $P(x)$ has real coefficients, hence if z is a non - real zero of $P(x)$, then also $P(\bar{z}) = 0$.

The product of zeros is equal to the constant term of $P(x)$ divided by the leading coefficient. Hence,

$$z \cdot \bar{z} \cdot 1 \cdot \alpha = \frac{13}{3} \Rightarrow$$

$$|z|^2 = \frac{13}{3\alpha} \Rightarrow -\sqrt{\frac{13}{3\alpha}} < |z| < \sqrt{\frac{13}{3\alpha}};\ 2 < \alpha < 3 \Rightarrow\ -\sqrt{\frac{13}{3 \cdot 3}} < |z| < \sqrt{\frac{13}{3 \cdot 2}}$$

$\Rightarrow -\sqrt{\dfrac{13}{9}} < |z| < \sqrt{\dfrac{13}{6}}$. The sum of the four roots is equal to the coefficient of x^3 multiplied by -1 and divided by the leading coefficient. Hence,

$$z + \bar{z} + 1 + \alpha = \frac{(-1)(-4)}{3} \Rightarrow\ 2\operatorname{Re} z = \frac{4}{3} - 1 - \alpha \Rightarrow \operatorname{Re} z = \frac{1}{6} - \frac{\alpha}{2};\ 2 < \alpha < 3 \Rightarrow$$

$$\frac{1}{6} - \frac{3}{2} < \operatorname{Re} z < \frac{1}{6} - \frac{2}{2} \Rightarrow -\frac{4}{3} < \operatorname{Re} z < -\frac{5}{6}.$$

2 Solution

(a) We use the division transformation and then seek partial fractions.

$$(x-1)(x-2) = x^2 - 3x + 2 \Rightarrow\ \frac{x^2}{(x-1)(x-2)} = 1 + \frac{3x-2}{(x-1)(x-2)}.$$

Let $\dfrac{3x-2}{(x-1)(x-2)} \equiv \dfrac{a}{x-1} + \dfrac{b}{x-2}$, a, b constants. Then $3x - 2 \equiv a(x-2) + b(x-1)$.

Putting $x = 1$ gives $a = -1$. Putting $x = 2$ gives $b = 4$.

Hence $\displaystyle\int \frac{x^2}{(x-1)(x-2)}dx = \int\left(1 - \frac{1}{x-1} + \frac{4}{x-2}\right)dx = \int dx - \int \frac{1}{x-1}dx + 4\int \frac{1}{x-2}dx =$

$$x - \ln|x-1| + 4\ln|x-2| =\ x + \ln\left|\frac{(x-2)^4}{x-1}\right|.$$

(b)

(i) $\int \cos^3 x dx = \int (1-\sin^2 x)(\sin x)' dx = \int (\sin x)' dx - \int \sin^2 x (\sin x)' dx = \sin x - \frac{\sin^3 x}{3}$.

(ii) $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int \sin^{-1} x (\sin^{-1} x)' dx = \frac{(\sin^{-1} x)^2}{2}$.

(c) Let $I = \int_0^1 \frac{\sqrt{x}}{1+x} dx$. Use the substitution $x = u^2 \Rightarrow dx = udu,\ x = 0 \Rightarrow u = 0$,

$x = 1 \Rightarrow u = 1$. Hence

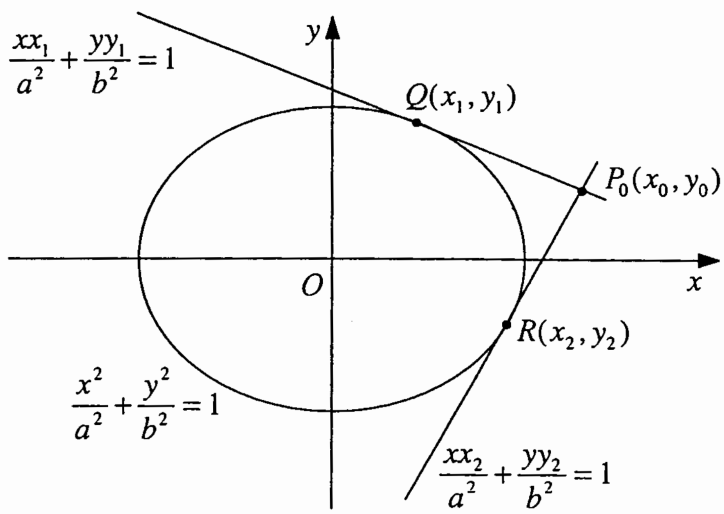
$I = \int_0^1 \frac{u}{1+u^2} 2udu = 2\int_0^1 \frac{u}{1+u^2} du = 2\int_0^1 \frac{(1+u^2)-1}{1+u^2} du = 2\int_0^1 du - 2\int_0^1 \frac{u}{1+u^2} du =$

$[2u]_0^1 - [2\tan^{-1} u]_0^1 = 2 - 2\cdot\frac{\pi}{4} = 2 - \frac{\pi}{2}$. Integrating by parts, $\int_0^1 \frac{1}{\sqrt{x}} \ln(1+x) dx =$

$[2\sqrt{x}\ln(1+x)]_0^1 - 2\int_0^1 \frac{\sqrt{x}}{1+x} dx = 2\ln 2 - 2I = 2\ln 2 - 4 + \pi$.

3 Solution

(a) Since $P_0(x_0, y_0)$ lies on the tangent P_0Q, then $\frac{x_0x_1}{a^2} + \frac{y_0y_1}{b^2} = 1$. Since $P_0(x_0, y_0)$ lies on the tangent P_0R, then $\frac{x_0x_2}{a^2} + \frac{y_0y_2}{b^2} = 1$. Hence both $Q(x_1, y_1)$ and $R(x_2, y_2)$ satisfy



$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$. But this is the equation of a straight line and is thus the equation of the chord of contact to tangents from $P_0(x_0, y_0)$.

(b) $x^2 + 2y^2 = 6 \Rightarrow \frac{x^2}{6} + \frac{y^2}{3} = 1$. The chord of contact to tangents from the point (4,–1) to the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ has equation $\frac{4x}{6} - \frac{y}{3} = 1 \Rightarrow 2x - y = 3$. Let

$T(x',y')$ be the extremity of the chord, then $2x'-y'=3 \Rightarrow y'=2x'-3$. Since the point $T(x',y')$ lies on the ellipse, then $x'^2+2y'^2=6$. Hence

$$x'^2+2(2x'-3)^2=6 \Rightarrow 9x'^2-24x'+12=0 \Rightarrow (3x'-2)(x'-2)=0.$$

Therefore the tangents to the ellipse $x^2+2y^2=6$ from the point $(4,-1)$ are $\frac{2}{3}x-\frac{10}{3}y=6 \Rightarrow x-5y=9$, with point of contact $T\left(\frac{2}{3},-\frac{5}{3}\right)$ and $2x+2y=6 \Rightarrow x+y=3$, with the point of contact $T(2,1)$.

4 Solution

(a) $z_1=4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \Rightarrow |z_1|=4, \arg z_1=\frac{\pi}{3}$.

$z_2=2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right) \Rightarrow |z_2|=2, \arg z_2=-\frac{\pi}{6}$.

(i) $|z_1^3|=|z_1|^3=64, \quad \arg(z_1^3)=3\arg z_1=\pi$.

(ii) $\left|\frac{1}{z_2}\right|=\frac{1}{|z_2|}=\frac{1}{2}, \quad \arg\left(\frac{1}{z_2}\right)=-\arg z_2=\frac{\pi}{6}$.

(iii) $\left|\frac{z_1^3}{z_2}\right|=|z_1^3|\cdot\left|\frac{1}{z_2}\right|=32, \quad \arg\left(\frac{z_1^3}{z_2}\right)=\arg(z_1^3)+\arg\left(\frac{1}{z_2}\right)=\frac{7\pi}{6}$. But $\frac{7\pi}{6}>\pi$. Therefore the principal argument of $\frac{z_1^3}{z_2}$ is $\frac{7\pi}{6}-2\pi=-\frac{5\pi}{6}$.

(b) (i) Using the triangle inequality, $|z+2|\le|z|+2=3$ and $|z+2|\ge 2-|z|=1$.

(ii) Since $z+2=2+\cos\theta+i\sin\theta$, then $\mathrm{Re}(z+2)>0 \Rightarrow \tan\arg(z+2)=\frac{\sin\theta}{2+\cos\theta}$.

Hence $\tan\arg(z+2)=\dfrac{2\tan\left(\frac{\theta}{2}\right)}{3+\tan^2\left(\frac{\theta}{2}\right)}$. When θ varies from $-\pi$ to π, $\frac{\theta}{2}$ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and, therefore, $\tan\left(\frac{\theta}{2}\right)$ varies from $-\infty$ to $+\infty$. Function $f(t)=\frac{2t}{3+t^2}$ on the interval $(-\infty,+\infty)$ has its minimum value $-\frac{1}{\sqrt{3}}$ and its maximum value $\frac{1}{\sqrt{3}}$.

Hence $-\dfrac{1}{\sqrt{3}} \le \tan\arg(z+2) \le \dfrac{1}{\sqrt{3}}$. Since $\mathrm{Re}(z+2) > 0$, then $-\dfrac{\pi}{6} \le \arg(z+2) \le \dfrac{\pi}{6}$.

(c) If x^2+1 is a factor of $P(x) = x^4 + px^3 + 2x + q$, then

$P(x) = (x^2+1)Q(x) \Rightarrow P(\pm i) = 0$;

$P(i) = 0 \Rightarrow 1 - ip + 2i + q = 0 \Rightarrow i(2-p) + (1+q) = 0$. As p and q are real, $p = 2$ and $q = -1$. Hence $P(x) = x^4 + 2x^3 + 2x - 1$.

Use the polynomial division

$$
\begin{array}{r l}
 & x^2+2x-1 \\
x^2+1\,\big) & x^4+2x^3+2x-1 \\
 & \underline{x^4+x^2} \\
 & 2x^3-x^2+2x-1 \\
 & \underline{2x^3+2x} \\
 & -x^2-1 \\
 & \underline{-x^2-1} \\
 & 0
\end{array}
$$

$\Rightarrow P(x) = (x^2+1)(x^2+2x-1)$. But $x^2+2x-1 = 0 \Rightarrow$ $x = -1 \pm \sqrt{2} \Rightarrow x^2+2x-1 = (x+1-\sqrt{2})(x+1+\sqrt{2})$.

Hence

$P(x) = (x^2+1)(x+1-\sqrt{2})(x+1+\sqrt{2})$ and this is the factorisation of $P(x)$ over **R**. As $x^2+1 = (x-i)(x+i)$, the factorisation of $P(x)$ over **C** is

$P(x) = (x-i)(x+i)(x+1-\sqrt{2})(x+1+\sqrt{2})$.

5 Solution

(a) $z = (6+5i)(7+2i) = (42-10) + i(12+35) = 32+47i$. Since $(6-5i)(7-2i) = \bar{z}$, then $(6-5i)(7-2i) = 32-47i$. It is clear $|z|^2 = |6+5i|^2|7+2i|^2$. Therefore we obtain $32^2+47^2 = (6^2+5^2)(7^2+2^2) = 61\cdot 53$.

(b) **(i)** α,β,γ satisfy x^3+2x-1. Hence $-\alpha,-\beta,-\gamma$ satisfy $(-x)^3+2(-x)-1=0 \Rightarrow$ $-x^3-2x-1=0 \Rightarrow x^3+2x+1=0$.

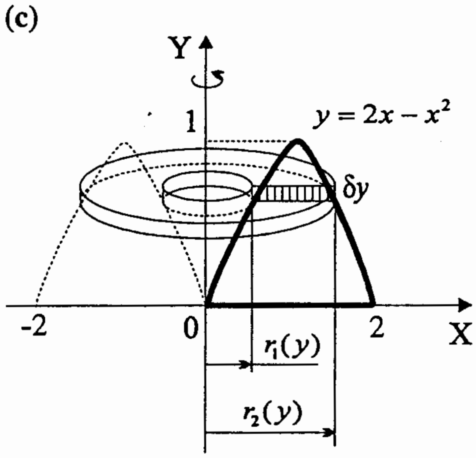
(ii) α,β,γ satisfy $x^3+2x-1=0$ and from (i) $-\alpha,-\beta,-\gamma$ satisfy $x^3+2x+1=0$. Hence $\alpha,-\alpha,\beta,-\beta,\gamma,-\gamma$ satisfy $(x^3+2x-1)(x^3+2x+1)=0$. Expanding, $x^6+4x^4+4x^2-1=0$.

(iii) α,β,γ satisfy $x^3+2x-1=0$. Hence $\alpha^2,\beta^2,\gamma^2$ satisfy $(x^{1/2})^3+2x^{1/2}-1=0$.

Rearrangements gives $x^{1/2}(x+2)=1$. Squaring and simplifying, $x(x+2)^2=1 \Rightarrow$

$x^3+4x^2+4x-1=0$.

(c)



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y)>r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of $y=2r-r^2$ considered as a quadratic equation. The slice has volume $\delta V=\pi(r_2+r_1)(r_2-r_1)\delta y$.

$$y=2r-r^2$$
$$r^2-2r+y=0$$
$$r_{1,2}=1\mp\sqrt{1-y}$$

$$r_2+r_1=2$$
$$r_2-r_1=2\sqrt{1-y}$$
$$\therefore\quad \delta V=4\pi\sqrt{1-y}\,\delta y.$$
$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{1}4\pi\sqrt{1-y}\,\delta y=\int_0^1 4\pi\sqrt{1-y}\,dy.$$

Substitution $y=1-y'$, $dy=-dy'$ gives

$$V=-4\pi\int_1^0\sqrt{y'}\,dy'=-4\pi\left[\frac{y'^{3/2}}{3/2}\right]_1^0=\frac{8\pi}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{8\pi}{3}$ cubic units.

6 Solution

(i) Choose $\downarrow$ as positive direction. Hence the equation of motion is $\ddot{x}=g-kv$.

Terminal velocity: as $\ddot{x}\to 0, v\to\left(\frac{g}{k}\right)^-$. Hence the terminal velocity $V=\frac{g}{k}$.

(ii) Choose $\uparrow$ as positive direction and initial position as origin.

Equation of motion: $\ddot{x}=-g-kv$.

Initial conditions: $x=0$, $v=V$.

Relation between x and v

$$v\frac{dv}{dx}=-(g+kv);$$

Relation between v and t

$$\frac{dv}{dt}=-(g+kv);$$

$$-dx = \frac{v}{g+kv}dv\,; \qquad -dt = \frac{dv}{g+kv}\,;$$

$$-dx = \frac{1}{k}\left(1-\frac{g}{g+kv}\right)dv\,; \qquad -dt = \frac{1}{k}\left(\frac{(g+kv)'}{g+kv}\right)dv\,;$$

$$-x + C = \frac{1}{k}\left\{v - \frac{g}{k}\ln(g+kv)\right\},\ C \text{ constant}; \qquad -t + A = \frac{1}{k}\ln(g+kv),\ A\,\text{constant};$$

$$x = 0, v = V \Rightarrow C = \frac{V}{k} - \frac{g}{k^2}\ln(g+kV) \Rightarrow$$

$$t = 0,\ v = V \Rightarrow\ A = \frac{1}{k}\ln(g+kV) \Rightarrow$$

$$x = \frac{g}{k^2}\ln\left(\frac{g+kv}{g+kV}\right) + \frac{V-v}{k}. \qquad (1) \qquad t = \frac{1}{k}\ln\left(\frac{g+kV}{g+kv}\right). \qquad (2)$$

From (2) $\dfrac{g+kV}{g+kv} = e^{kt} \Rightarrow \dfrac{\frac{g}{k}+V}{\frac{g}{k}+v} = e^{kt}$. From (i) $\dfrac{g}{k} = V \Rightarrow \dfrac{2V}{v+V} = e^{kt} \Rightarrow$

$$v + V = 2Ve^{-kt} \Rightarrow v = V\left(2e^{-kt} - 1\right). \qquad (3)$$

From (1) $x = \dfrac{1}{k}\left(\dfrac{g}{k}\right)\ln\left(\dfrac{\frac{g}{k}+v}{\frac{g}{k}+V}\right) + \dfrac{(V-v)}{k}$. As $\dfrac{g}{k} = V \Rightarrow x = \dfrac{V}{k}\ln\left(\dfrac{V+v}{2V}\right) + \dfrac{(V-v)}{k} \Rightarrow$

$$x = \frac{V}{k}\ln\left(\frac{1}{2} + \frac{v}{2V}\right) + \frac{(V-v)}{k}\,; \qquad (4)$$

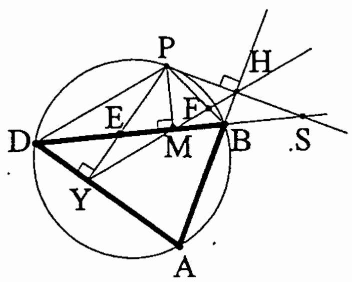
$v = 0, x = H \Rightarrow H = \dfrac{V}{k}\ln\dfrac{1}{2} + \dfrac{V}{k} \Rightarrow H = \dfrac{V}{k}(1 - \ln 2)$. Using (3) from (4)

$$x = \frac{V}{k}\ln\left(\frac{1}{2} + \frac{2e^{-kt}-1}{2}\right) + \frac{V}{k} - \frac{V}{k}\left(2e^{-kt}-1\right) \Rightarrow\ x = \frac{V}{k}\ln\left(e^{-kt}\right) + 2\frac{V}{k} - \frac{V}{k}2e^{-kt} \Rightarrow$$

$$x = \frac{V}{k}\left(2 - 2e^{-kt} - kt\right).$$

7 Solution

We first prove a preliminary result. Lemma. Δ ABD be inscribed in a circle. P is a point on a minor arc BD. H, Y and M are the feet of the perpendiculars from P to AB produced, AD and BD respectively. Then Δ PYH is similar to Δ PDB.



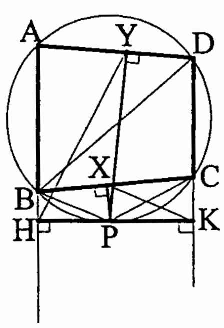
Proof. Note the known fact, that the points Y, M and H are collinear. (The line YH is the so called Simpson line, see Specimen paper 6, *N* 7 (b)).

We show that $\angle PHY = \angle PBD$ and $\angle PYH = \angle PDB$. Let PH=DB meet at S.

Triangles PMS and BHS are similar to each other, as they are rectangular and have a common angle $\angle PSD$. Hence $\frac{HS}{BS} = \frac{MS}{PS} \Rightarrow$ the triangles PBS and MHS are similar, as they also have a common angle $\angle PSD$. From here $\angle PBS = \angle MHS$ and hence $\angle PHY = \angle PBD$.

Analogously the right-angular triangles PME and DYE are similar, as $\angle PEM = \angle DEY \Rightarrow$ the triangles MEY and PED are similar, and hence $\angle EYM = \angle PDE$.

(i) In order to show that Δ XPK is similar to Δ HPY, we prove that $\angle HPY = \angle XPK$ and $\angle YHP = \angle KXP$. The first step: $\angle HPY = \angle XPK$. The sum of angles of any quadrilateral is $360°$. Consider the quadrilaterals XPKC and YAHP. Every of these quadrilaterals has two right angles, hence $\angle XPK + \angle XCK = 180°$ and $\angle YPH + \angle YAH = 180°$. But $\angle DCB + \angle XCK = 180°$. Hence $\angle XPK = \angle DCB$. ADCB is a cyclic quadrilateral, so



$\angle DAB + \angle DCB = 180°$ (as opposite angles) $\Rightarrow$

$\angle DAB + \angle XPK = 180°$. So we have $\angle YPH + \angle YAH = 180°$ and

$\angle DAB + \angle XPK = 180° \Rightarrow \angle YPH = \angle XPK$.

The second step: $\angle YHP = \angle KXP$. It is sufficient to show that

$\angle YHP + \angle XKP + \angle YPH = 180°$. By lemma $\angle YHP = \angle DBP$ and

$\angle XKP = \angle XCP$. Consider a cyclic quadrilateral DBPC. $\angle DBP + \angle DCP = 180°$.

But $\angle DCP = \angle DCB + \angle XCP = 180° \Rightarrow \angle DBP + \angle DCB + \angle XCP = 180°$. A quadrilateral ABCD is a cyclic $\Rightarrow \angle DCB = 180° - \angle DAB$. Hence

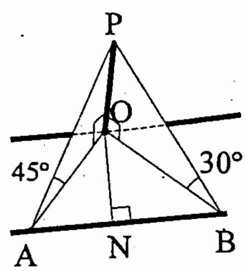
$\angle DBP + (180° - \angle DAB) + \angle XCP = 180°$

$\Rightarrow \angle DBP + \angle XCP = \angle DAB$. But $\angle DAB = 180° - \angle YPH$, as the quadrilateral AHPY has two right angles. Hence $\angle DBP + \angle XCP = 180 - \angle YPH$. As we saw $\angle DBP = \angle YHP$, we have $\angle YHP + \angle XCP + \angle YPH = 180°$.

(ii) Δ XPK is similar to Δ HPY, hence $\frac{PX}{PH} = \frac{PK}{PY} \Rightarrow PX \cdot PY = PH \cdot PK$. Also

$\frac{PX}{PH} = \frac{XK}{HY}$ and $\frac{PK}{PY} = \frac{XK}{HY}$, multiplying these equalities, $\frac{PX}{PH} \cdot \frac{PK}{PY} = \frac{(XK)^2}{(HY)^2}$.

8 Solution



$OP = 2\,m$, $\angle AOB = 150°$

(a) **(i)** OP is a vertical pole $\Rightarrow$ the triangle AOP is rectangular

$\Rightarrow AO = OP \cdot \cot \angle PAO \Rightarrow AO = 2 \cdot \cot 45 \Rightarrow AO = 2\,m$

Analogously, $\angle POB = 90° \Rightarrow BO = OP \cdot \cot \angle PBO \Rightarrow$

$BO = 2 \cdot \cot 30° \Rightarrow BO = 2\sqrt{3}\,m$.

Consider the triangle AOB. By the well known theorem.

$AB^2 = AO^2 + BO^2 - 2AO \cdot BO \cos \angle AOB \Rightarrow$

$AB^2 = 2^2 + \left(2\sqrt{3}\right)^2 - 2 \cdot 2 \cdot 2\sqrt{3} \cdot \cos 150° \Rightarrow AB^2 = 4 + 12 + 8\sqrt{3} \cdot \cos 30° \Rightarrow$

$AB^2 = 16 + 8\sqrt{3} \cdot \frac{\sqrt{3}}{2} \Rightarrow AB^2 = 28 \Rightarrow AB = 2\sqrt{7}$.

(ii) Let ON be the perpendicular produced from O to AB. If S is the area of the triangle AOB, then $S = \frac{AB \cdot ON}{2}$ *and* $S = \frac{OA \cdot OB}{2} \sin \angle AOB$. Equating,

$\frac{AB \cdot ON}{2} = \frac{OA \cdot OB}{2} \sin \angle AOB \Rightarrow$

$ON = \frac{OA \cdot OB}{AB} \sin \angle AOB$; $OA = 2, OB = 2\sqrt{3}, AB = 2\sqrt{7}, \angle AOB = 150° \Rightarrow$

$ON = \frac{2 \cdot 2\sqrt{3}}{2\sqrt{7}} \cdot \sin 30° \Rightarrow ON = \frac{2\sqrt{3}}{\sqrt{7}} \cdot \frac{1}{2} \Rightarrow ON = \frac{\sqrt{3}}{\sqrt{7}}\,m$ is the width of the canal.

(b) Every of six lines intersects with another five lines. Thus there are five points of intersection on every line and every point belongs exactly to two lines. Hence on the whole there are $\frac{6\cdot 5}{2}=15$ points of intersection.

(i) There exist 15 different points, 6 lines, and each line contains 5 points. An elementary outcome is a set of three points. The total number of all elementary outcomes is equal to $\binom{15}{3}=\frac{15!}{12!3!}=5\cdot 7\cdot 13$. Success is when all points lie on the one line. The number of ways in which one can choose three points from the set of 5 lying on each line is equal to $\binom{5}{3}$.

The amount of lines is 6, hence the number of success is 6 times greater, that is,

$6\cdot\binom{5}{3}=6\cdot 2\cdot 5$. The desired probability is equal to $\frac{6\cdot\binom{5}{3}}{\binom{15}{3}}=\frac{6\cdot 2\cdot 5}{5\cdot 7\cdot 13}=\frac{12}{91}$.

(ii) Let A be an event that 4 points don't lie on one line. Analogously one is able to calculate a probability of the opposite event $\overline{\mathrm{A}}$, that is $P(\overline{\mathrm{A}})=\frac{6\cdot\binom{5}{4}}{\binom{15}{4}}$.

As we get $P(\mathrm{A})+(\overline{\mathrm{A}})=1$, we get $P(\mathrm{A})=1-\frac{6\cdot\binom{5}{4}}{\binom{15}{4}}=1-\frac{4}{14\cdot 13}=1-\frac{2}{91}=\frac{89}{91}$.

Specimen Paper 6.

1 Solution

(a) (i) Let us show for the curve $x^2y^2 - x^2 + y^2 = 0$ that $|y| \le 1$ and $|y| \le |x|$. Since $y^2 = \frac{x^2}{x^2+1}$, we get $|y| = \frac{|x|}{\sqrt{x^2+1}} \le |x|$ or $|y| = \frac{|x|}{\sqrt{x^2+1}} \le 1$ because $\sqrt{x^2+1} \ge 1$ and $\sqrt{x^2+1} \ge |x|$.

(ii) The graph of $x^2y^2 - x^2 + y^2 = 0$ is represented by two curves $y = \frac{x}{\sqrt{x^2+1}}$ and $y = -\frac{x}{\sqrt{x^2+1}}$. Each graph has point symmetry about the origin (it suffices to construct the graph for $x \ge 0$), as both functions are odd. It is easily seen that

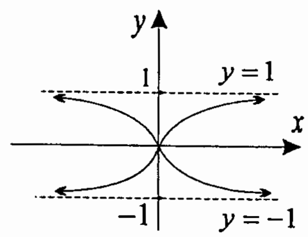
$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \pm 1, \qquad \lim_{x\to\pm\infty} \frac{-x}{\sqrt{x^2+1}} = \mp 1.$$

Hence we can deduce that the equations of the asymptotes are $y = \pm 1$.

The tangent to the curve $y = f(x)$ which passes through the point $(x_0, y_0 = f(x_0))$ is given by $y = k(x - x_0) + y_0$, where $k = f'(x_0)$ is the gradient of the tangent.

If $y = f(x) = \pm\frac{x}{\sqrt{x^2+1}}$, and $f(0) = 0$,

$$f'(x) = \pm\frac{1}{\sqrt{x^2+1}} \mp \frac{x^2}{\sqrt{(x^2+1)^3}}, \quad f'(0) = \pm 1,$$



we get the following equation of the tangents at the origin:

$y = \pm x$.

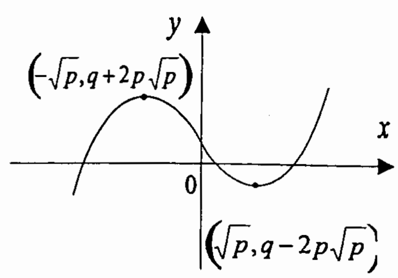
(b) (i) Let us sketch the graph of $y = x^3 - 3px + q$, where $p > 0$, and q is a real number. It is easily seen that $y \to -\infty$ as $x \to -\infty$, and $y \to +\infty$ as $x \to +\infty$.

Find the turning points:

$y'(x) = 3x^2 - 3p = 0 \Rightarrow x = \pm\sqrt{p}$,

$y(\sqrt{p}) = p^{3/2} - 3p^{3/2} + q = q - 2p^{3/2}$,

$y(-\sqrt{p}) = -p^{3/2} + 3p^{3/2} + q = q + 2p^{3/2}$.



Hence

$y(-\sqrt{p}) = q + 2p\sqrt{p}$ is the point of maximum,

$y(\sqrt{p}) = q - 2p\sqrt{p}$ is the point of minimum.

(ii) It is clear that the roots of the equation $x^3 - 3px + q = 0$ are all real if and only if the following system of equations holds

$$\begin{cases} q + 2p^{3/2} > 0 \\ q - 2p^{3/2} < 0 \end{cases},$$

that is, the point of maximum must be above the x-axis, and the point of minimum is to be situated below the x-axis. Hence $-2p^{3/2} < q < 2p^{3/2} \Rightarrow |q| < 2p^{3/2} \Rightarrow q^2 < 4p^3$.

2 Solution

(a) $\int \frac{4}{x^2+2x-1}dx = 4\int \frac{1}{(x+1)^2-2}dx = 4\int \left\{\frac{1}{x+1-\sqrt{2}} - \frac{1}{x+1+\sqrt{2}}\right\}\cdot \frac{1}{2\sqrt{2}}dx =$

$\sqrt{2}\int \frac{1}{x+1-\sqrt{2}}dx - \sqrt{2}\int \frac{1}{x+1+\sqrt{2}}dx = \sqrt{2}\ln\left|x+1-\sqrt{2}\right| - \sqrt{2}\ln\left|x+1+\sqrt{2}\right| =$

$\sqrt{2}\ln\left|\frac{x+1-\sqrt{2}}{x+1+\sqrt{2}}\right|$.

(b) Let $I = \int\limits_{1/a}^{a} \frac{\ln x}{x^2+1}dx$; $u = \frac{1}{x} \Rightarrow x = \frac{1}{u} \Rightarrow dx = \frac{-1}{u^2}du$,

$x = \frac{1}{a} \Rightarrow u = a,\ x = a \Rightarrow u = \frac{1}{a}$.

Hence $I = \int\limits_{a}^{1/a} \frac{\ln\left(\frac{1}{u}\right)}{\frac{1}{u^2}+1}\cdot\left(\frac{-1}{u^2}\right)du = -\int\limits_{a}^{1/a} \frac{\ln\left(\frac{1}{u}\right)}{1+u^2}du = \int\limits_{1/a}^{a} \frac{\ln\left(\frac{1}{u}\right)}{1+u^2}du = -\int\limits_{1/a}^{a} \frac{\ln(u)}{1+u^2}du = -I \Rightarrow$

$I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$.

(c) Integrating by parts, $I_n = \int\limits_0^1 \left(1-x^2\right)^n dx = \left[x\left(1-x^2\right)\right]_0^1 - \int\limits_0^1 xn(-2x)\left(1-x^2\right)^{n-1}dx =$

$2n\int\limits_0^1 x^2\left(1-x^2\right)^{n-1}dx = -2n\int\limits_0^1 \left(1-x^2-1\right)\left(1-x^2\right)^{n-1}dx =$

$-2n\int\limits_0^1 \left(1-x^2\right)^n dx + 2n\int\limits_0^1 \left(1-x^2\right)^{n-1}dx =$

$-2nI_n + 2nI_{n-1}$. Hence $I_n = -2nI_n + 2nI_{n-1} \Rightarrow I_n = \dfrac{2n}{2n+1} I_{n-1}$.

To prove that $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$ for $n \geq 1$, we use the induction.

Define the statement $S(n)$: $I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$, $n \geq 1$.

Consider $S(1)$: $n=1$, $I_1 = \int_0^1 (1-x^2)dx = \left[x - \dfrac{x^3}{3}\right]_0^1 = \dfrac{2}{3}$, but

$\left.\dfrac{2^{2n}(n!)^2}{(2n+1)!}\right|_{n=1} = \dfrac{2^2}{3!} = \dfrac{2}{3} \Rightarrow S(1)$ is true. Let k be a positive integer. If $S(k)$ is true, then

$I_k = \dfrac{2^{2k}(k!)^2}{(2k+1)!}$ for $n = k \geq 1$.

Consider

$S(k+1)$.

If $S(k)$ is true, we get $I_{k+1} = \dfrac{2(k+1)}{2(k+1)+1} I_k = \dfrac{2(k+1)}{2k+3} \cdot \dfrac{2^{2k}(k!)^2}{(2k+1)!} =$

$$I_{k+1} = \frac{2(k+1)(2k+2)}{(2k+3)(2k+2)} \cdot \frac{2^{2k}(k!)^2}{(2k+1)!} = \frac{2^2(k+1)^2 \cdot 2^{2k}(k!)^2}{(2k+3)!} = \frac{2^{2k+2}((k+1)!)^2}{(2(k+1)+1)!} =$$

$$\frac{2^{2(k+1)}((k+1)!)^2}{(2(k+1)+1)!}.$$

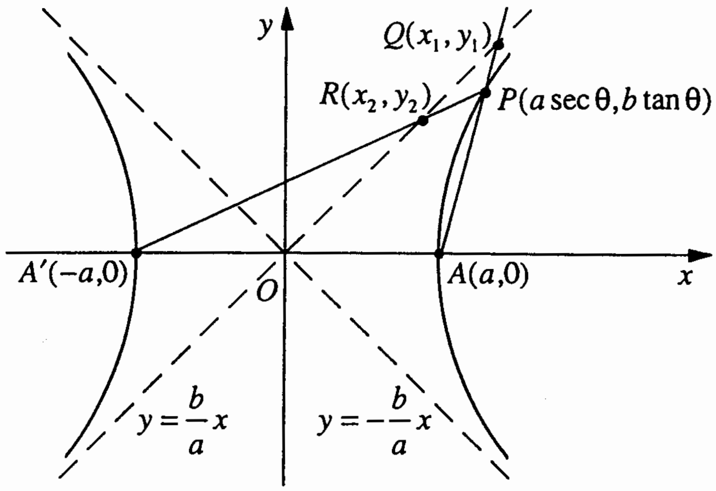
Hence for all positive integers k, $S(k)$ true implies that $S(k+1)$ is true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers n.

$I_n = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$ for $n \geq 1$.

3 Solution

(i) The line AP has equation

$$y = \frac{b\tan\theta}{a(\sec\theta - 1)}(x - a).$$



Since the point Q lies on the line AP, then $y_1 = \frac{b\tan\theta}{a(\sec\theta - 1)}(x_1 - a)$.

Since the point Q lies on the asymptote $y = \frac{b}{a}x$, then $y_1 = \frac{b}{a}x_1$. Therefore

$$x_1 = \frac{\tan\theta}{(\sec\theta - 1)}(x_1 - a) \Rightarrow x_1 = \frac{a\tan\theta}{\tan\theta - \sec\theta + 1} = \frac{a\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$ and

$$y_1 = \frac{b\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}.$$ Thus the point Q has coordinates $\left(\frac{a\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}, \frac{b\cos\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right)$.

Similarly the line $A'P$ has equation $y = \frac{b\tan\theta}{a(\sec\theta + 1)}(x + a)$. Since the point R lies on the line $A'P$, then $y_2 = \frac{b\tan\theta}{a(\sec\theta + 1)}(x_2 + a)$.

Since the point R lies on the asymptote $y = \frac{b}{a}x$, then $y_2 = \frac{b}{a}x_2$. So

$$x_2 = \frac{\tan\theta}{(\sec\theta + 1)}(x_2 + a) \Rightarrow x_2 = \frac{-a\tan\theta}{\tan\theta - \sec\theta - 1} = \frac{a\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$ and

$$y_2 = \frac{b\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}.$$ Thus the point R has coordinates $\left(\frac{a\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}, \frac{b\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}\right)$.

(ii)

$$QR^2=(x_1-x_2)^2+(y_1-y_2)^2=\frac{a^2\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}+\frac{b^2\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}=a^2+b^2.$$

Thus the length of QR is $\sqrt{a^2+b^2}$ and hence is independent of θ. The area of the triangle PQR is $\frac{1}{2}\cdot QR\cdot h$ where h is the height of the triangle. Since h is the distance from $P(a\sec\theta, b\tan\theta)$ to the line $y=\frac{b}{a}x$, then $h=\frac{\left|\frac{b}{a}\cdot a\sec\theta-b\tan\theta\right|}{\sqrt{\left(\frac{b}{a}\right)^2+1}}=\frac{ba|\sec\theta-\tan\theta|}{\sqrt{a^2+b^2}}$. Therefore the area of the triangle PQR is

$$\frac{1}{2}\cdot\sqrt{a^2+b^2}\cdot\frac{ba|\sec\theta-\tan\theta|}{\sqrt{a^2+b^2}}=\frac{1}{2}ab|\sec\theta-\tan\theta|.$$

4 Solution

(a) If $\alpha,\beta,$ and γ satisfy $x^3+px+q=0$, then $\alpha^2,\beta^2,$ and γ^2 satisfy

$\left(x^{1/2}\right)^3+px^{1/2}+q=0$.

Rearrangement gives $x^{1/2}(x+p)=-q$.

Squaring gives $x(x+p)^2=q^2$.

Simplifying, we get $x^3+2px^2+p^2x-q^2=0$.

(b) Let $z=cos\,q+i\,sin\,q$. Then by *DeMoivre's* theorem, $z^4=\cos 4\theta+i\sin 4\theta$. But by the Binomial theorem, $z^4=\sum_{k=0}^{4}\binom{4}{k}i^k\sin^k\theta\cos^{4-k}\theta$. Equating real and imaginary parts,

$$\cos 4\theta=\binom{4}{0}\cos^4\theta+\binom{4}{2}\left(-\sin^2\theta\right)\cos^2\theta+\binom{4}{4}\sin^4\theta,$$

$$\sin 4\theta=\binom{4}{1}\sin\theta\cos^3\theta+\binom{4}{3}\left(-\sin^3\theta\right)\cos\theta.$$

Hence

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta}{\cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta} \Rightarrow \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}. \quad (1)$$

(i) $\tan 4\theta = 1 \Rightarrow 4\theta = \tan^{-1} 1 + \pi n$, n integer, $\Rightarrow 4\theta = \frac{\pi}{4} + \pi n$, $n = 0, \pm 1, \pm 2, \ldots \Rightarrow$

$$\theta = \frac{\pi(4n+1)}{16}, \ n \text{ integral.} \quad (2)$$

(ii) Let $\tan\theta = x$. Then from (1)

$$\tan 4\theta = 1 \Leftrightarrow \frac{4x - 4x^3}{1 - 6x^2 + x^4} = 1 \Leftrightarrow x^4 + 4x^3 - 6x^2 - 4x + 1 = 0.$$

Hence from (2) $x = \tan\frac{\pi(4n+1)}{16}$, $n = 0, \pm 1, \pm 2, \ldots$ But of these there are only four distinct non-zero values: $\alpha = \tan\frac{\pi}{16}$, $\beta = \tan\frac{5}{16}\pi$, $\gamma = -\tan\frac{3}{16}\pi$, $\delta = -\tan\frac{7}{16}\pi$.

α, β, γ and δ are the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$. Hence, $\sum \alpha = -4$ and

$$\sum \alpha\beta = -6 \Rightarrow \left(\sum \alpha\right)^2 = (-4)^2 \Rightarrow \sum \alpha^2 + 2\sum \alpha\beta = 16 \Rightarrow \sum \alpha^2 = 16 - 2\cdot(-6).$$

So $\left(\alpha^2 + \beta^2 + \gamma^2 + \delta^2\right) = 28$ and hence $\tan^2\frac{\pi}{16} + \tan^2\frac{5}{16}\pi + \tan^2\frac{3}{16}\pi + \tan^2\frac{7}{16}\pi = 28$.

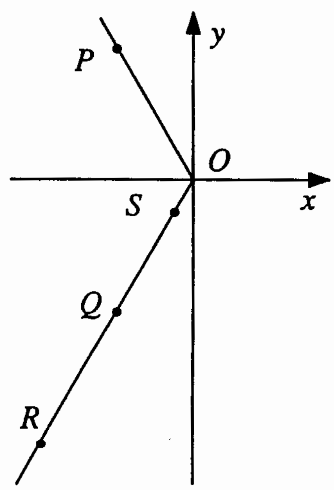
5 Solution

(a) (i) $z = -1 + i\sqrt{3} = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$.

(ii) $\bar{z} = 2\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$,

$z^2 = 4\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$,

$\frac{1}{z} = \frac{1}{2}\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$.



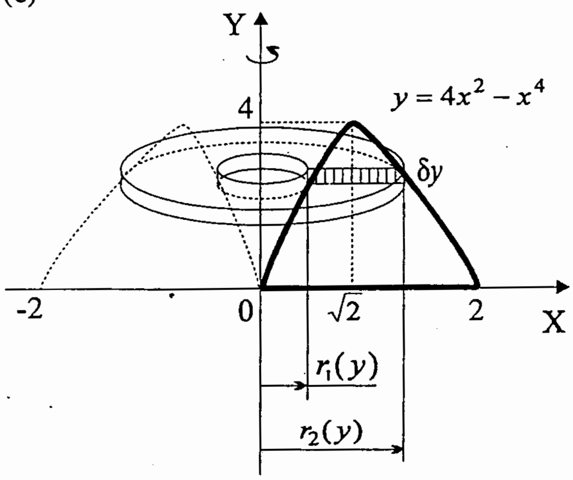
(b) **(i, ii)** Let $P(z)=z^2+(2+i)z+k$;

$P(1-2i)=0 \Rightarrow (1-2i)^2+(2+i)(1-2i)+k=0 \Rightarrow k=-1+7i$.

If $z=1-2i$ and w is another root of $P(z)$, then

$$z\cdot w=k \Rightarrow w=\frac{-1+7i}{1-2i} \Rightarrow w=-3+i.$$

(c)



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y)>r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of $y=4r^2-r^4$ considered as a biquadratic equation. The slice has volume $\delta V=\pi\left(r_2^2-r_1^2\right)\delta y$.

$$y=4r^2-r^4$$
$$r^2=z$$
$$z^2-4z+y=0$$

$$z_{1,2}=2\mp\sqrt{4-y}$$
$$r_{1,2}=\sqrt{2\mp\sqrt{4-y}}$$
$$\therefore \quad \delta V=2\pi\sqrt{4-y}\,\delta y.$$
$$\therefore \quad V=\lim_{\delta y\to 0}\sum_{y=0}^{4}2\pi\sqrt{4-y}\,\delta y=2\pi\int_0^4\sqrt{4-y}\,dy.$$

Substitution $y=4-y'$, $dy=-dy'$ gives

$$V=-2\pi\int_4^0\sqrt{y'}\,dy'=-2\pi\left[\frac{y'^{3/2}}{3/2}\right]_4^0=\frac{32\pi}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{32\pi}{3}$ cubic units.

6 Solution

Choose $\uparrow$ as a positive direction. Choose the center of the earth as origin.

Equation of motion: $\ddot{x}=\dfrac{-gR^2}{x^2}$.

Initial conditions: $t=0,\ x=R,\ v=V$.

(i) Relation between x and v: $v\dfrac{dv}{dx}=-\dfrac{gR^2}{x^2} \Rightarrow vdv=-\dfrac{gR^2}{x^2}dx \Rightarrow \dfrac{v^2}{2}+C=\dfrac{gR^2}{x}$, C constant.

$x = R, v = V \Rightarrow C = -\frac{V^2}{2} + gR$. As $V = \sqrt{gR}$, we have

$C = \frac{gR}{2} \Rightarrow \frac{v^2}{2} = \frac{gR^2}{x} - \frac{gR}{2} \Rightarrow$

$$v = \sqrt{gR}\sqrt{\frac{2R-x}{x}}. \qquad (1)$$

(ii) $v = 0 \Rightarrow$ from (1) $x = 2R$. Hence the projectile reaches the height R above the surface of the earth.

From the equation of motion $\frac{dv}{dt} = -\frac{gR^2}{x^2}$. (2)

But from (1) $x = \frac{2gR^2}{gR+v^2}$. Substituting this into (2), $\frac{dv}{dt} = -gR^2\left(\frac{gR+v^2}{2gR^2}\right)^2 \Rightarrow$

$\frac{dv}{dt} = -\frac{\left(gR+v^2\right)^2}{4gR^2} \Rightarrow \frac{-dt}{4gR^2} = \frac{dv}{\left(gR+v^2\right)^2} \Rightarrow \frac{-t}{4gR^2} + A = \int \frac{dv}{\left(gR+v^2\right)^2}$, A constant.

Let us calculate $I = \int \frac{dv}{\left(a^2+v^2\right)^2}$, $a = \sqrt{gR}$; $\int \frac{dv}{\left(a^2+v^2\right)^2} = \frac{1}{a^4}\int \frac{1}{\left(1+\left(\frac{v}{a}\right)^2\right)^2}dv =$

$\frac{1}{a^3}\int \frac{(v/a)'}{\left(1+\left(\frac{v}{a}\right)^2\right)^2}dv$.

Let $u = \frac{v}{a} \Rightarrow I = \frac{1}{a^3}\int \frac{du}{\left(1+u^2\right)^2}$.

But

$\int \frac{du}{\left(1+u^2\right)^2} = \int \frac{1+u^2-u^2}{\left(1+u^2\right)^2}du = \int \frac{1}{1+u^2}du - \int \frac{u^2}{\left(1+u^2\right)^2}du = \tan^{-1}u - \int \frac{u}{2}\frac{\left(u^2\right)'}{\left(1+u^2\right)^2}du =$

$\tan^{-1}u + \int \frac{u}{2}\left(\frac{1}{1+u^2}\right)' du =$

$\tan^{-1}u + \frac{u}{2}\cdot\frac{1}{1+u^2} - \frac{1}{2}\int \frac{1}{1+u^2}du = \tan^{-1}u + \frac{u}{2\left(1+u^2\right)} - \frac{1}{2}\tan^{-1}u \Rightarrow$

$$I=\frac{1}{a^3}\left\{\frac{1}{2}\tan^{-1}u+\frac{u}{2\left(1+u^2\right)}\right\};$$

$$u=\frac{v}{a}\Rightarrow I=\frac{1}{2a^3}\tan^{-1}\frac{v}{a}+\frac{1}{a^3}\cdot\frac{v/a}{2\left(1+\frac{v^2}{a^2}\right)}\Rightarrow$$

$$I=\frac{1}{2a^3}\tan^{-1}\frac{v}{a}+\frac{v}{2a^2\left(a^2+v^2\right)},\quad a=\sqrt{gR}\,.$$

$$\text{Hence }\frac{-t}{4gR^2}+A=\frac{1}{2(gR)^{3/2}}\tan^{-1}\frac{v}{\sqrt{gR}}+\frac{v}{2gR\left(gR+v^2\right)};\qquad (3)$$

$$t=0,\ v=\sqrt{gR}\Rightarrow\ A=\frac{1}{2(gR)^{3/2}}\tan^{-1}1+\frac{1}{2\sqrt{gR}\,(gR+gR)}\Rightarrow$$

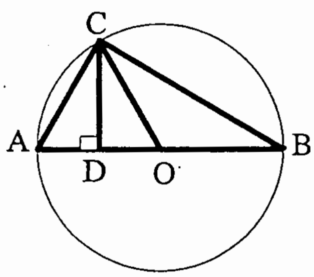
$$A=\frac{\pi}{4}\cdot\frac{1}{2(gR)^{3/2}}+\frac{1}{4(gR)^{3/2}}\Rightarrow$$

$$A=\frac{1}{4(gR)^{3/2}}\cdot\left(\frac{\pi}{2}+1\right).$$

$$v=0\Rightarrow\ \text{from (3) } t=A\cdot 4gR^2\Rightarrow t=\left(\frac{\pi}{2}+1\right)\cdot\frac{4gR^2}{4(gR)\sqrt{gR}}\Rightarrow t=\left(\frac{\pi}{2}+1\right)\cdot\sqrt{\frac{R}{g}}\,.$$

7 Solution

(a) Let O be the center of the circle, and r be its radius



(i) Consider the rectangular triangles ACB and CDB. These triangles have the common angle $\angle$CAB. Hence ΔACB is similar to ΔCDB. From here

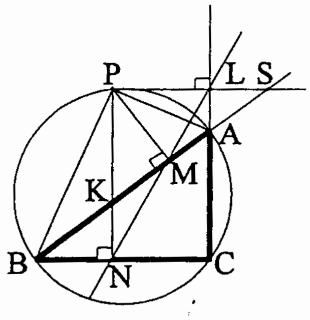
$$\frac{\mathrm{AD}}{\mathrm{CD}}=\frac{\mathrm{CD}}{\mathrm{BD}}\Rightarrow \mathrm{CD}^2=\mathrm{AD}\cdot\mathrm{BD};$$

$$\mathrm{AD}=a,\ \mathrm{BD}=b\Rightarrow \mathrm{CD}=\sqrt{ab}\,.$$

(ii) $\mathrm{AB}=2r\Rightarrow r=\frac{a+b}{2}$; $\mathrm{OC}=r$ and the triangle CDO is rectangular

$$\Rightarrow \mathrm{OC}>\mathrm{CD}\Rightarrow\ r>\sqrt{ab}\Rightarrow\ \frac{a+b}{2}>\sqrt{ab}\,.$$

(b) In order to prove that L, M and N are collinear, it is sufficient to show that $\angle LMA = \angle NMB$. For this purpose we show, that $\angle NMB = \angle BPN = \angle SPA = \angle LMA$.



The first step: $\angle NMB = \angle BPN$. The triangles PEM and BEN are right-angular and $\angle PEM = \angle BEN \Rightarrow \Delta$ PEM is similar to Δ BEN $\Rightarrow \frac{BE}{PE} = \frac{NE}{ME}$. But $\angle PEB = \angle MEN \Rightarrow \Delta$ PEB is similar to Δ MEN $\Rightarrow \angle NMB = \angle BPN$.

The second step: $\angle BPN = \angle SPA$. The point P lies on the circle $\Rightarrow$ PACB is a cyclic quadrilateral $\Rightarrow \angle PAC + \angle PBC = 180°$. But $\angle PAC + \angle PAL = 180°$. Hence $\angle PBC = \angle PAL$. From here, as the triangles PNB and PLA are right-angular, we have Δ PNB is similar to Δ PLA $\Rightarrow \angle BPN = \angle APL$.

The third step: $\angle SPA = \angle LMA$. It is obvious that Δ ALS is similar to Δ PMS, as these right-angular triangles have the common angle $\angle PSM$. Hence $\frac{PS}{AS} = \frac{MS}{LS} \Rightarrow$ Δ MLS is similar to Δ PAS $\Rightarrow \angle SPA = \angle LMA$.

8 Solution

(a) $\left(a+\frac{1}{a}\right)^2 = \left(a-\frac{1}{a}\right)^2 + 4 \Rightarrow \left(a+\frac{1}{a}\right)^2 \geq 4 \Rightarrow a+\frac{1}{a} \geq 2$ (equality iff $a = \frac{1}{a}$, i.e. $a=1$);

$$a^2+\frac{1}{a^2}-\left(a+\frac{1}{a}\right) = \left(a^2-a\right)+\left(\frac{1}{a^2}-\frac{1}{a}\right) = a(a-1)-\frac{1}{a^2}(a-1) = (a-1)\left(a-\frac{1}{a^2}\right) =$$

$$(a-1)\frac{\left(a^3-1\right)}{a^2} = (a-1)(a-1)\frac{\left(a^2+a+1\right)}{a^2} = (a-1)^2\frac{\left(a^2+a+1\right)}{a^2} \geq 0 \Rightarrow a^2+\frac{1}{a^2} \geq a+\frac{1}{a}$$

with equality iff $a=1$.

(b) (i) If α, β are roots of $x^2-x+1=0$, then $\alpha+\beta=1$ and $\alpha\beta=1$ by *Vieta's* theorem. Hence $(\alpha+\beta)^2=1 \Rightarrow \alpha^2+\beta^2=1-2\alpha\beta \Rightarrow \alpha^2+\beta^2=-1$,

$A_n = \alpha^n+\beta^n = \alpha^{n-1}\cdot\alpha+\beta^{n-1}\cdot\beta$,

$\alpha+\beta=1 \Rightarrow \alpha=1-\beta$ and $\beta=1-\alpha$. Hence $A_n=\alpha^{n-1}(1-\beta)+\beta^{n-1}(1-\alpha) \Rightarrow$

$A_n=\alpha^{n-1}+\beta^{n-1}-\alpha^{n-1}\beta-\beta^{n-1}\alpha \Rightarrow \ A_n=A_{n-1}-\alpha\beta\left(\alpha^{n-2}+\beta^{n-2}\right)$; $\alpha\beta=1 \Rightarrow$

$A_n=A_{n-1}-A_{n-2}$ for $n \geq 3$.

(ii) Define the statement $S(n)$: $A_n=2\cos\frac{n\pi}{3}$ for $n \geq 1$.

Consider $S(1)$: $n=1,\ A_1=2\cos\frac{\pi}{3}=1 \Rightarrow S(1)$ is true.

Consider $S(2)$: $n=2,\ A_2=2\cos\frac{2\pi}{3}=-1 \Rightarrow S(2)$ is true.

Let k be a positive integer, $k \geq 2$. If $S(n)$ is true for all integer $n \leq k$, then

$$A_n=2\cos\frac{n\pi}{3}\ ,\ n=1,2,3,\ldots k.$$

Consider $S(k+1)$. If $S(n)$ is true for $n=1,2,3\ldots k$, we get $A_{k+1}=A_k-A_{k-1}=$

$2\cos\frac{k\pi}{3}-2\cos\frac{(k-1)\pi}{3}=2\left(\cos\frac{k\pi}{3}-2\cos\frac{(k-1)\pi}{3}\right)$.

But $\cos a-\cos b=2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b-a}{2}\right)$, $a,b \in \mathbf{R}$.

Hence $A_{k+1}=4\sin\left(\frac{(2k-1)}{2}\cdot\frac{\pi}{3}\right)\sin\left(-\frac{1}{2}\cdot\frac{\pi}{3}\right) \Rightarrow \ A_{k+1}=-4\sin\frac{\pi}{6}\sin\left(\left(k-\frac{1}{2}\right)\frac{\pi}{3}\right) \Rightarrow$

$A_{k+1}=-2\sin\left((k+1)\frac{\pi}{3}-\frac{\pi}{2}\right) \Rightarrow \ A_{k+1}=2\cos(k+1)\frac{\pi}{3}$.

Hence for $k \geq 2$, $S(n)$ true for all positive integers $n \leq k$ implies $S(k+1)$ is true. But

$S(1)$,

$S(2)$ are true.

Therefore by induction, $S(n)$ is true for all positive integers n:

$A_n = 2\,cos\frac{np}{3}\ for\, n\ ^3\ 1.$