7SD Solutions Series

Worked Solutions to Popular Mathematics Texts

Suggested Worked Solutions to

"4 Unit Mathematics"

(Text book for the NSW HSC by D. Arnold and G. Arnold)

Chapter 6
Volumes

Solutions prepared by: Michael M. Yastreboff and Dr Victor V. Zalipaev

7SD Solutions Series

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Solutions are to “4 Unit Mathematics”

[by D. Arnold and G. Arnold (1993), ISBN 0 340 54335 3]

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The 7SD team welcomes all feedback.

Correspondence should be addressed to:

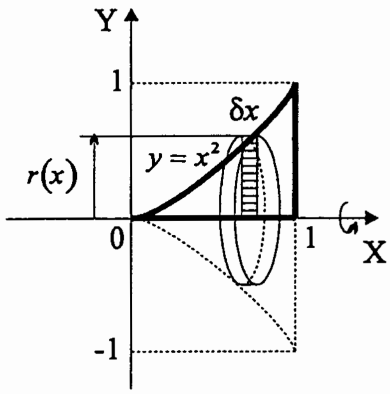
7SD attn: Michael Yastreboff

PO Box 123

Kensington NSW 2033

Exercise 6.1

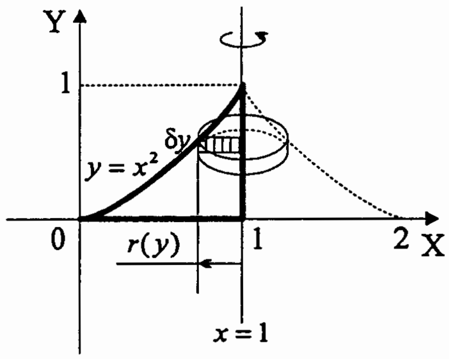
1 Solution



a) A slice taken perpendicular to the axis of rotation is a disk of thickness δx and radius $r(x) = x^2$. The slice has volume $\delta V = \pi x^4 \delta x$.

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \pi x^4 \delta x = \int_0^1 \pi x^4 dx = \left.\frac{\pi x^5}{5}\right|_0^1 = \frac{\pi}{5}.$$

$\therefore$ the volume of the solid is $\frac{\pi}{5}$ cubic units.

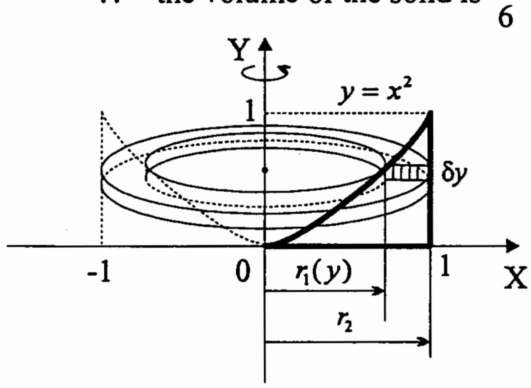


b) A slice taken perpendicular to the axis of rotation is a disk of thickness δy and radius $r(y) = 1 - \sqrt{y}$. The slice has volume $\delta V = \pi(1 - \sqrt{y})^2 \delta y$.

$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{1} \pi(1-\sqrt{y})^2 \delta y = \int_0^1 \pi(1-\sqrt{y})^2 dy = \int_0^1 \pi(1 - 2\sqrt{y} + y)dy$$

$$= \pi\left(y - \frac{2y^{3/2}}{3/2} + \frac{y^2}{2} \right)\Bigg|_0^1 = \frac{\pi}{6}.$$

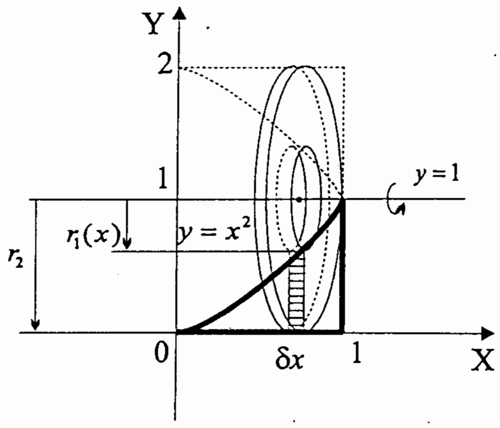
$\therefore$ the volume of the solid is $\frac{\pi}{6}$ cubic units.



c) A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y) = \sqrt{y}$ and $r_2 = 1$. The slice has volume $\delta V = \pi(r_2^2 - r_1^2)\delta y = \pi(1 - y)\delta y$.

$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{1} \pi(1-y)\delta y = \int_0^1 \pi(1-y)dy = \pi\left(y - \frac{y^2}{2} \right)\Bigg|_0^1 = \frac{\pi}{2}.$$

$\therefore$ the volume of the solid is $\dfrac{\pi}{2}$ cubic units.



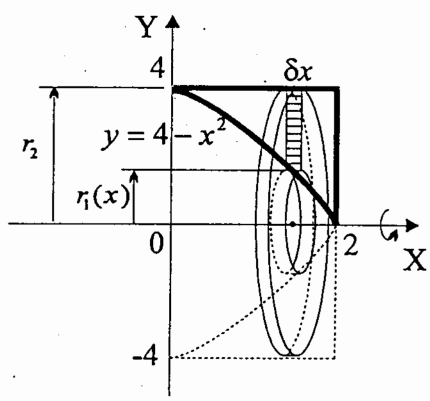
d) A slice taken perpendicular to the axis of rotation is an annulus of thickness δx with radii $r_1(x) = 1 - x^2$ and $r_2 = 1$. The slice has volume

$$\delta V = \pi\left(r_2^2 - r_1^2\right)\delta x = \left[1 - \left(1 - x^2\right)^2\right]\delta x = \pi\left(2x^2 - x^4\right)\delta x$$

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \pi\left(2x^2 - x^4\right)\delta x = \int_0^1 \pi\left(2x^2 - x^4\right)dx = \pi\left(\frac{2x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{7\pi}{15}.$$

$\therefore$ the volume of the solid is $\dfrac{7\pi}{15}$ cubic units.

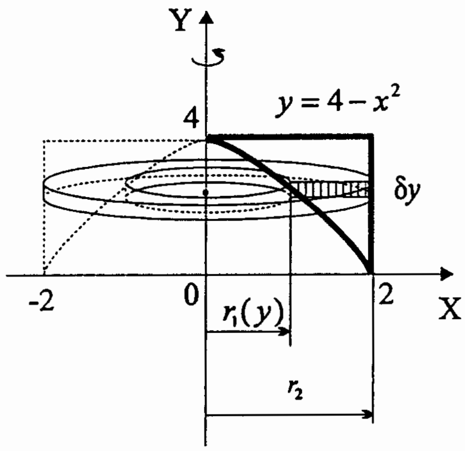
2 Solution



a) A slice taken perpendicular to the axis of rotation is an annulus of thickness δx with radii $r_1(x) = 4 - x^2$ and $r_2 = 4$. The slice has volume $\delta V = \pi\left(r_2^2 - r_1^2\right)\delta x = \pi\left(8x^2 - x^4\right)\delta x$.

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{2} \pi\left(8x^2 - x^4\right)\delta x = \int_0^2 \pi\left(8x^2 - x^4\right)dx = \pi\left(\frac{8x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^2 = \frac{224\pi}{15}.$$

$\therefore$ the volume of the solid is $\dfrac{224\pi}{15}$ cubic units.

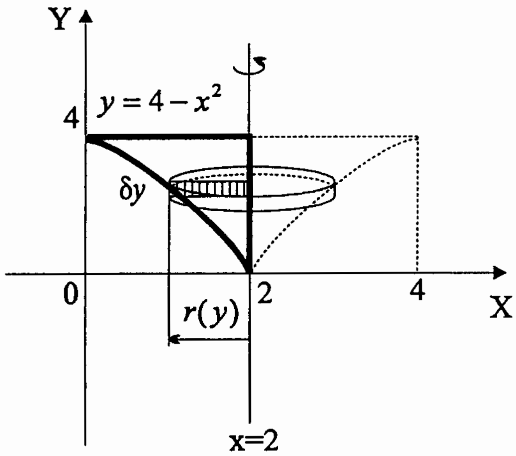


b) A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)=\sqrt{4-y}$ and $r_2=2$. The slice has volume

$$\delta V=\pi\left(r_2^2-r_1^2\right)\delta y=\left[2^2-\left(\sqrt{4-y}\right)^2\right]\delta y=\pi y\delta y.$$

$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{4}\pi y\delta y=\int_0^4\pi y dy=\pi\frac{y^2}{2}\bigg|_0^4=8\pi.$$

$\therefore$ the volume of the solid is 8π cubic units.



c) A slice taken perpendicular to the axis of rotation is a disk of thickness δy and radius $r(y)=2-\sqrt{4-y}$. The slice has volume

$$\delta V=\pi\left(2-\sqrt{4-y}\right)^2\delta y=\pi\left(8-y-4\sqrt{4-y}\right)\delta y$$

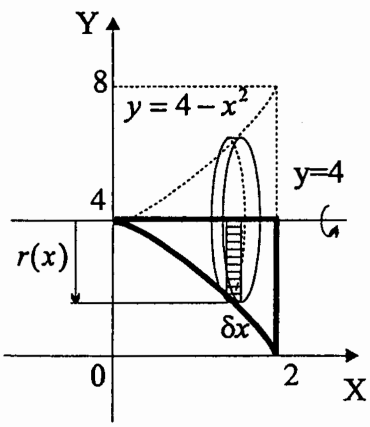
.

$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{4}\pi\left(8-y-4\sqrt{4-y}\right)\delta y=\int_0^4\pi\left(8-y-4\sqrt{4-y}\right)dy.$$

Substitution $y=4-y',\quad dy=-dy'$ gives

$$V=-\int_4^0\pi\left(4+y'-4\sqrt{y'}\right)dy'=-\pi\left(4y'+\frac{y'^2}{2}-\frac{4y'^{3/2}}{3/2}\right)\bigg|_4^0=\frac{8\pi}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{8\pi}{3}$ cubic units.

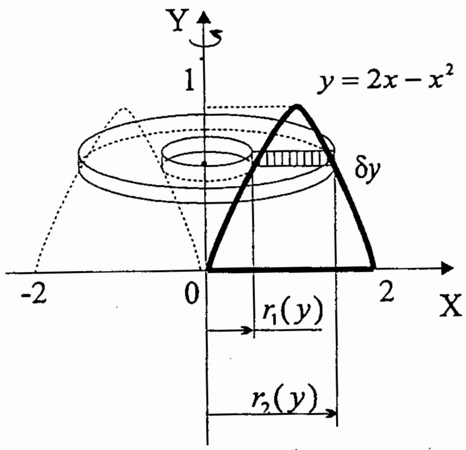


d) A slice taken perpendicular to the axis of rotation is a disk of thickness δx and radius $r(x)=x^2$. The slice has volume $\delta V=\pi x^4\delta x$.

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{2}\pi x^4\delta x=\int_0^2\pi x^4 dx=\frac{\pi x^5}{5}\bigg|_0^2=\frac{32\pi}{5}.$$

$\therefore$ the volume of the solid is $\dfrac{32\pi}{5}$ cubic units.

3 Solution



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y) > r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of $y = 2r - r^2$ considered as a quadratic equation. The slice has volume $\delta V = \pi(r_2 + r_1)(r_2 - r_1)\delta y$.

$$y = 2r - r^2$$
$$r^2 - 2r + y = 0$$
$$r_{1,2} = 1 \mp \sqrt{1-y}$$
$$r_2 + r_1 = 2$$

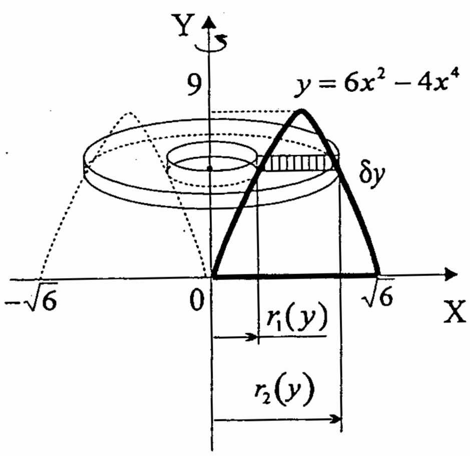
$$r_2 - r_1 = 2\sqrt{1-y}$$
$$\therefore \quad \delta V = 4\pi\sqrt{1-y}\,\delta y$$
$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{1} 4\pi\sqrt{1-y}\,\delta y = \int_0^1 4\pi\sqrt{1-y}\,dy.$$

Substitution $y = 1 - y'$, $dy = -dy'$ gives

$$V = -4\pi\int_1^0 \sqrt{y'}\,dy' = -4\pi\frac{y'^{3/2}}{3/2}\Bigg|_1^0 = \frac{8\pi}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{8\pi}{3}$ cubic units.

4 Solution



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y) > r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of $y = 6r^2 - r^4$ considered as a biquadratic equation. The slice has volume $\delta V = \pi(r_2^2 - r_1^2)\delta y$.

$$y = 6r^2 - r^4$$
$$r^4 - 6r^2 + y = 0$$
$$z = r^2$$
$$z^2 - 6z + y = 0$$

$$z_{1,2} = 3 \mp \sqrt{9-y}$$
$$r_1 = \sqrt{z_1} = \sqrt{3 - \sqrt{9-y}}$$
$$r_2 = \sqrt{z_2} = \sqrt{3 + \sqrt{9-y}}.$$
$$\therefore \quad \delta V = \pi(r_2^2 - r_1^2)\delta y = 2\pi\sqrt{9-y}\,\delta y.$$

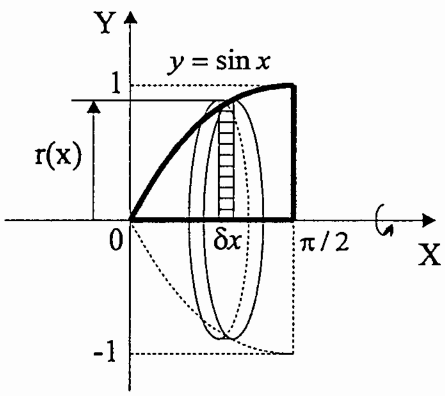
$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{9} 2\pi\sqrt{9-y}\,\delta y = \int_0^9 2\pi\sqrt{9-y}\,dy .$$

Substitution $y = 9 - y'$, $dy = -dy'$ gives

$$V = -2\pi\int_9^0 \sqrt{y'}\,dy' = -2\pi\frac{y'^{3/2}}{3/2}\Bigg|_9^0 = 36\pi .$$

$\therefore$ the volume of the solid is 36π cubic units.

5 Solution

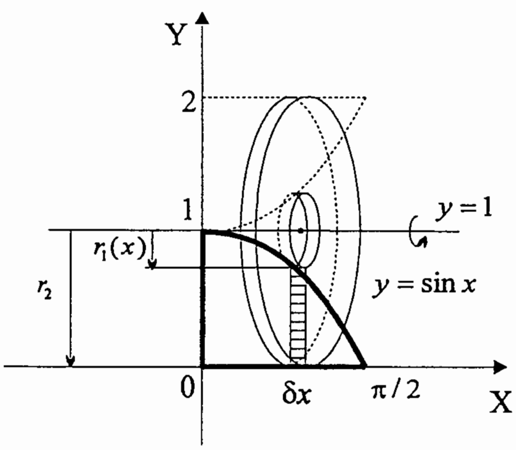


A slice taken perpendicular to the axis of rotation is a disk of thickness δx and radius $r(x) = \sin x$. The slice has volume
$\delta V = \pi r^2(x)\delta x = \pi \sin^2 x\,\delta x$.

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi/2} \pi \sin^2 x\,\delta x = \int_0^{\pi/2} \pi \sin^2 x\,dx = \pi\int_0^{\pi/2} \frac{1-\cos 2x}{2}dx = \frac{\pi}{2}\left(x - \frac{\sin 2x}{2}\right)\Bigg|_0^{\pi/2} = \frac{\pi^2}{4}.$$

$\therefore$ the volume of the solid is $\dfrac{\pi^2}{4}$ cubic units.

6 Solution



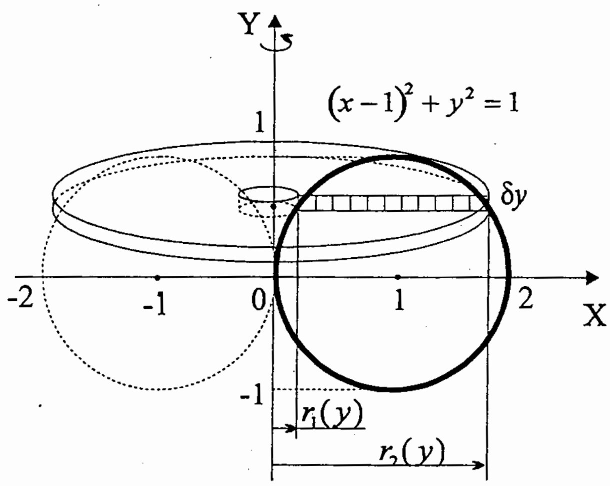
A slice taken perpendicular to the axis of rotation is an annulus of thickness δx with radii $r_1(x) = 1 - \cos x$ and $r_2 = 1$. The slice has volume
$\delta V = \pi\left(r_2^2 - r_1^2\right)\delta x = \pi\left(2\cos x - \cos^2 x\right)\delta x$.

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi/2} \pi\left(2\cos x - \cos^2 x\right)\delta x = \int_0^{\pi/2} \pi\left(2\cos x - \cos^2 x\right)dx$$

$$= \int_0^{\pi/2} \pi\left(2\cos x - \frac{1+\cos 2x}{2}\right)dx = \pi\left(2\sin x - \frac{x}{2} - \frac{\sin 2x}{4}\right)\Bigg|_0^{\pi/2} = 2\pi - \frac{\pi^2}{4}.$$

$\therefore$ the volume of the solid is $2\pi - \frac{\pi^2}{4}$ cubic units.

7 Solution



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y) > r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of $(r-1)^2 + y^2 = 1$ considered as a quadratic equation. The slice has volume

$$\delta V = \pi(r_2 + r_1)(r_2 - r_1)\delta y.$$

$$(r-1)^2 + y^2 = 1$$

$$r^2 - 2r + y^2 = 0$$

$$r_{1,2} = 1 \mp \sqrt{1-y^2}$$

$$r_2 + r_1 = 2$$

$$r_2 - r_1 = 2\sqrt{1-y^2}$$

$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=-1}^{1} 4\pi\sqrt{1-y^2}\,\delta y = \int_{-1}^{1} 4\pi\sqrt{1-y^2}\,dy.$$

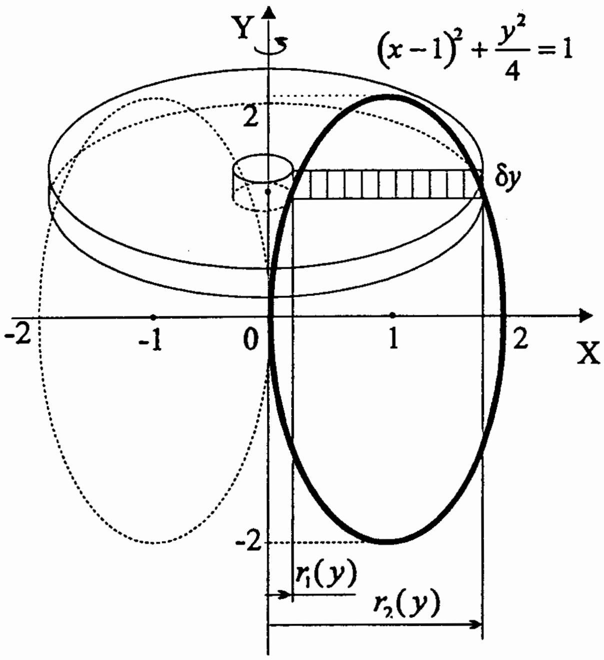
Substitution $y = \sin\varphi$, $dy = \cos\varphi\, d\varphi$ gives

$$V = 4\pi \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2\varphi}\,\cos\varphi\, d\varphi = 4\pi \int_{-\pi/2}^{\pi/2} \cos^2\varphi\, d\varphi = 4\pi \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\varphi}{2}\, d\varphi$$

$$= 2\pi\left(\varphi + \frac{\sin 2\varphi}{2}\right)\Bigg|_{-\pi/2}^{\pi/2} = 2\pi^2.$$

$\therefore$ the volume of the solid is $2\pi^2$ cubic units.

8 Solution



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y)>r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of $(r-1)^2+\frac{y^2}{4}=1$ considered as a quadratic equation. The slice has volume $\delta V=\pi(r_2+r_1)(r_2-r_1)\delta y$.

$$(r-1)^2+\frac{y^2}{4}=1$$

$$r^2-2r+\frac{y^2}{4}=0$$

$$r_{1,2}=1\mp\sqrt{1-\frac{y^2}{4}}$$

$$r_2+r_1=2$$

$$r_2-r_1=2\sqrt{1-\frac{y^2}{4}}$$

$$\therefore\quad \delta V=2\pi\sqrt{4-y^2}\,\delta y.$$

$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=-2}^{2}2\pi\sqrt{4-y^2}\,\delta y=\int_{-2}^{2}2\pi\sqrt{4-y^2}\,dy.$$

Substitution $y=2\sin\varphi$, $dy=2\cos\varphi\,d\varphi$ gives

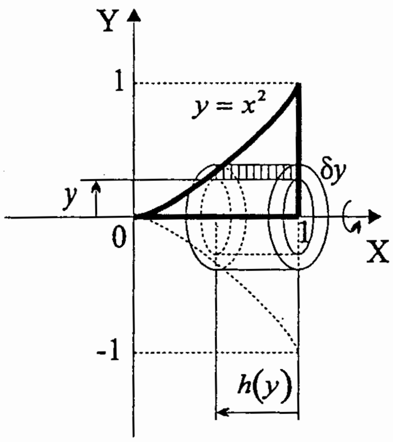
$$V=4\pi\int_{-\pi/2}^{\pi/2}\sqrt{4-4\sin^2\varphi}\,\cos\varphi\,d\varphi=8\pi\int_{-\pi/2}^{\pi/2}\cos^2\varphi\,d\varphi=8\pi\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\varphi}{2}\,d\varphi$$

$$=4\pi\left(\varphi+\frac{\sin 2\varphi}{2}\right)\Bigg|_{-\pi/2}^{\pi/2}=4\pi^2.$$

$\therefore$ the volume of the solid is $4\pi^2$ cubic units.

Exercise 6.2

1 Solution



a) The typical cylindrical shell has radii y, $y + \delta y$, and height $h(y) = 1 - \sqrt{y}$. This shell has volume

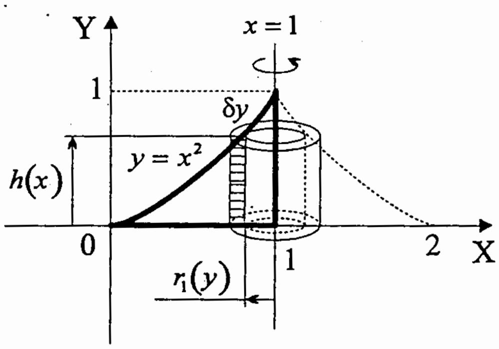
$$\delta V = \pi\left[(y+\delta y)^2 - y^2\right]h(y) = 2\pi\left(1-\sqrt{y}\right)y\,\delta y$$

(ignoring $(\delta y)^2$).

$$\therefore\quad V = \lim_{\delta y \to 0}\sum_{y=0}^{1} 2\pi\left(1-\sqrt{y}\right)y\,\delta y = 2\pi\int_0^1 \left(1-\sqrt{y}\right)y\,dy$$

$$= 2\pi\left(\frac{y^2}{2} - \frac{y^{5/2}}{5/2}\right)\Bigg|_0^1 = \frac{\pi}{5}.$$

$\therefore$ the volume of the solid is $\dfrac{\pi}{5}$ cubic units.



b) The typical cylindrical shell has radii $r_1(x) = 1 - x$, $r_2(x) = 1 - x + \delta x$, and height $h(x) = x^2$. This shell has volume

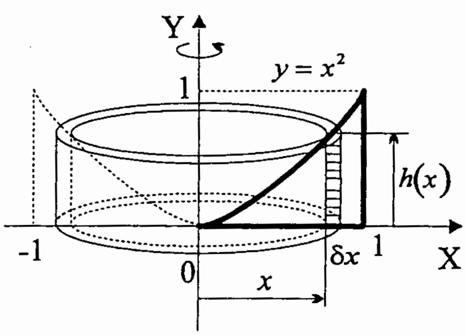
$$\delta V = \pi\left[(1-x+\delta x)^2 - (1-x)^2\right]h(x)$$

$$= 2\pi x^2(1-x)\delta x \text{ (ignoring } (\delta x)^2\text{).}$$

$$\therefore\quad V = \lim_{\delta x \to 0}\sum_{x=0}^{1} 2\pi x^2(1-x)\delta x$$

$$= 2\pi\int_0^1 x^2(1-x)\,dx = 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{\pi}{6}.$$

$\therefore$ the volume of the solid is $\dfrac{\pi}{6}$ cubic units.



c) The typical cylindrical shell has radii x, $x + \delta x$, and height $h(x) = x^2$. This shell has volume

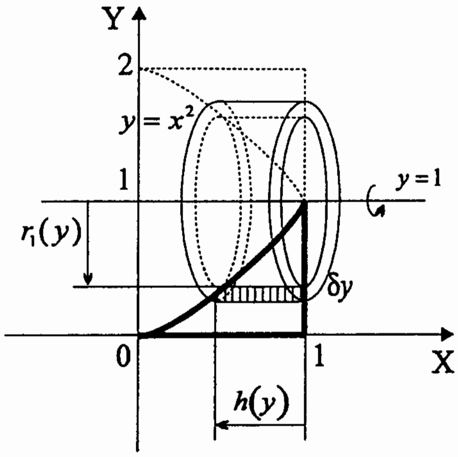
$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]h(x) = 2\pi x^3\,\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore\quad V = \lim_{\delta x \to 0}\sum_{x=0}^{1} 2\pi x^3\,\delta x = 2\pi\int_0^1 x^3\,dx$$

$$= 2\pi\frac{x^4}{4}\Bigg|_0^1 = \frac{\pi}{2}.$$

$\therefore$ the volume of the solid is $\dfrac{\pi}{2}$ cubic units.



d) The typical cylindrical shell has radii $r_1(y) = 1 - y$, $r_2(y) = 1 - y + \delta y$, and height $h(y) = 1 - \sqrt{y}$. This shell has volume

$$\delta V = \pi\left[(1 - y + \delta y)^2 - (1 - y)^2\right]h(y)$$
$$= 2\pi(1 - y)\left(1 - \sqrt{y}\right)\delta y$$

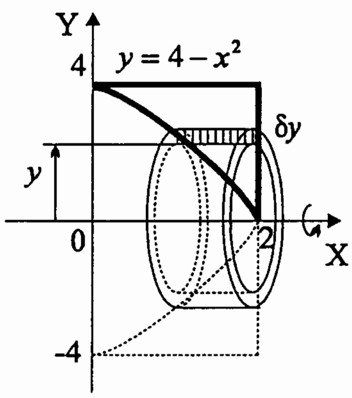
(ignoring $(\delta y)^2$).

$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{1} 2\pi(1 - y)\left(1 - \sqrt{y}\right)\delta y$$
$$= 2\pi\int_0^1 (1 - y)\left(1 - \sqrt{y}\right)dy$$

$$= 2\pi\int_0^1 \left(1 - y^{1/2} - y + y^{3/2}\right)dy = 2\pi\left(y - \frac{y^{3/2}}{3/2} - \frac{y^2}{2} + \frac{y^{5/2}}{5/2}\right)\Bigg|_0^1 = \frac{7\pi}{15}.$$

$\therefore$ the volume of the solid is $\dfrac{7\pi}{15}$ cubic units.

2 Solution



a) The typical cylindrical shell has radii y, $y + \delta y$, and height $h(y) = 2 - \sqrt{4 - y}$. This shell has volume

$$\delta V = \pi\left[(y + \delta y)^2 - y^2\right]h(y) = 2\pi\left(2 - \sqrt{4 - y}\right)y\,\delta y$$

(ignoring $(\delta y)^2$).

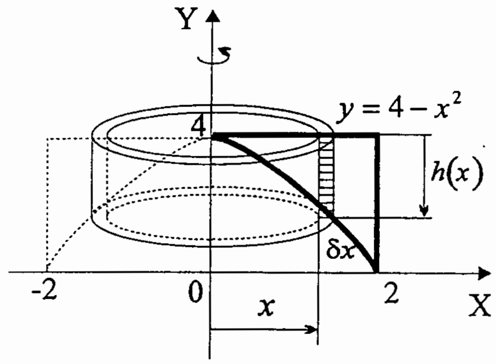
$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{4} 2\pi\left(2 - \sqrt{4 - y}\right)y\,\delta y$$
$$= 2\pi\int_0^4 \left(2 - \sqrt{4 - y}\right)y\,dy.$$

Substitution $y = 4 - y'$, $dy = -dy'$ gives

$$V = -2\pi\int_4^0 (4 - y')\left(2 - \sqrt{y'}\right)dy' = 2\pi\int_0^4 \left(8 - 4y'^{1/2} - 2y' + y'^{3/2}\right)dy'$$

$$= 2\pi\left(8y' - \frac{4y'^{3/2}}{3/2} - y'^2 + \frac{y'^{5/2}}{5/2}\right)\Bigg|_0^4 = \frac{224\pi}{15}.$$

$\therefore$ the volume of the solid is $\dfrac{224\pi}{15}$ cubic units.



b) The typical cylindrical shell has radii x, $x+\delta x$, and height $h(x)=x^2$. This shell has volume

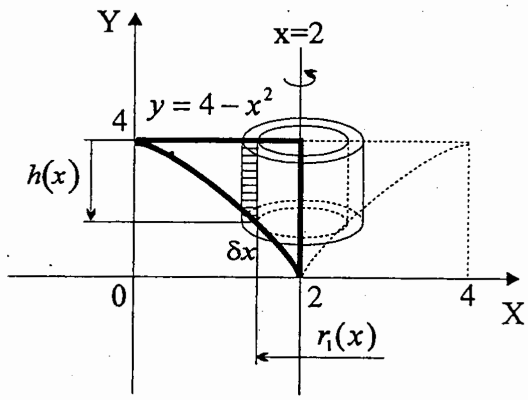
$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)=2\pi x^3\,\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore \quad V=\lim_{\delta x\to 0}\sum_{x=0}^{2}2\pi x^3\,\delta x=2\pi\int_0^2 x^3\,dx$$

$$=2\pi\left.\frac{x^4}{4}\right|_0^2=8\pi .$$

$\therefore$ the volume of the solid is 8π cubic units.



c) The typical cylindrical shell has radii $r_1(x)=2-x$, $r_2(x)=2-x+\delta x$, and height $h(x)=x^2$. This shell has volume

$$\delta V=\pi\left[(2-x+\delta x)^2-(2-x)^2\right]h(x)$$

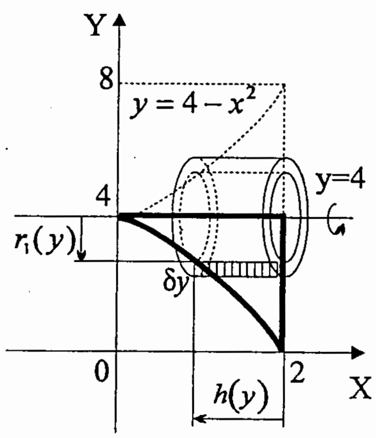
$$=2\pi x^2(2-x)\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore \quad V=\lim_{\delta x\to 0}\sum_{x=0}^{2}2\pi x^2(2-x)\delta x$$

$$=2\pi\int_0^2 x^2(2-x)\,dx=2\pi\left.\left(2\frac{x^3}{3}-\frac{x^4}{4}\right)\right|_0^2=\frac{8\pi}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{8\pi}{3}$ cubic units.



d) The typical cylindrical shell has radii $r_1(y)=4-y$, $r_2(y)=4-y+\delta y$, and height $h(y)=2-\sqrt{4-y}$. This shell has volume

$$\delta V=\pi\left[(4-y+\delta y)^2-(4-y)^2\right]h(y)$$

$$=2\pi(4-y)\left(2-\sqrt{4-y}\right)\delta y \ \text{(ignoring } (\delta y)^2\text{)}.$$

$$\therefore \quad V=\lim_{\delta y\to 0}\sum_{y=0}^{4}2\pi(4-y)\left(2-\sqrt{4-y}\right)\delta y$$

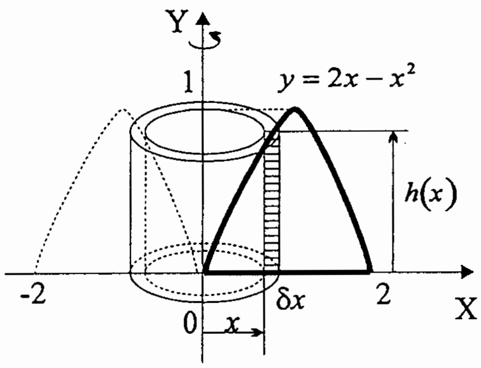
$$=2\pi\int_0^4 (4-y)\left(2-\sqrt{4-y}\right)dy .$$

Substitution $y=4-y'$, $dy=-dy'$ gives

$$V=-2\pi\int_4^0 y'\left(2-\sqrt{y'}\right)dy'=2\pi\left.\left(2\cdot\frac{y^2}{2}-\frac{y^{5/2}}{5/2}\right)\right|_0^4=\frac{32\pi}{5}.$$

$\therefore$ the volume of the solid is $\dfrac{32\pi}{5}$ cubic units.

3 Solution



The typical cylindrical shell has radii x, $x + \delta x$, and height $h(x) = 2x - x^2$. This shell has volume

$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]h(x) = 2\pi x(2x - x^2)\delta x$$

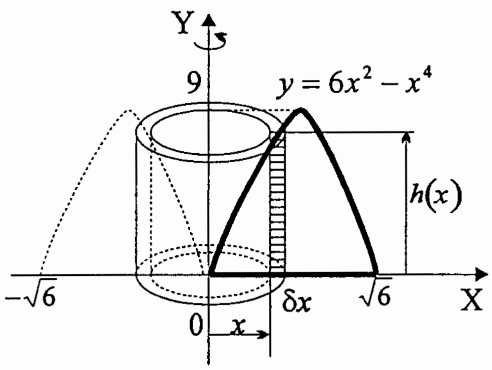
(ignoring $(\delta x)^2$).

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x(2x - x^2)\delta x$$

$$= 2\pi \int_0^2 x(2x - x^2)dx = 2\pi\left(2 \cdot \frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^2 = \frac{8\pi}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{8\pi}{3}$ cubic units.

4 Solution



The typical cylindrical shell has radii x, $x + \delta x$, and height $h(x) = 6x^2 - x^4$. This shell has volume

$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]h(x) = 2\pi x(6x^2 - x^4)\delta x$$

(ignoring $(\delta x)^2$).

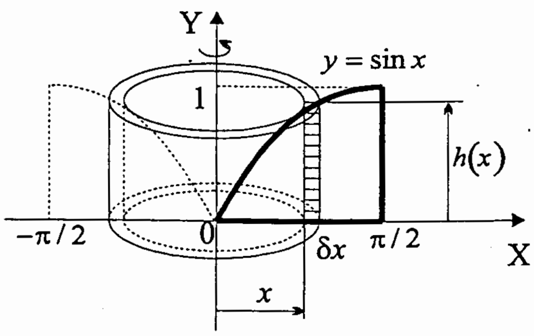
$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{\sqrt{6}} 2\pi x(6x^2 - x^4)\delta x$$

$$= 2\pi \int_0^{\sqrt{6}} x(6x^2 - x^4)dx$$

$$= 2\pi \int_0^{\sqrt{6}} x(6x^2 - x^4)dx = 2\pi\left(6 \cdot \frac{x^4}{4} - \frac{x^6}{6}\right)\Bigg|_0^{\sqrt{6}} = 36\pi.$$

$\therefore$ the volume of the solid is 36π cubic units.

5 Solution



The typical cylindrical shell has radii x, $x+\delta x$, and height $h(x)=\sin x$. This shell has volume

$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]h(x) = 2\pi x \sin x\,\delta x$$

(ignoring $(\delta x)^2$).

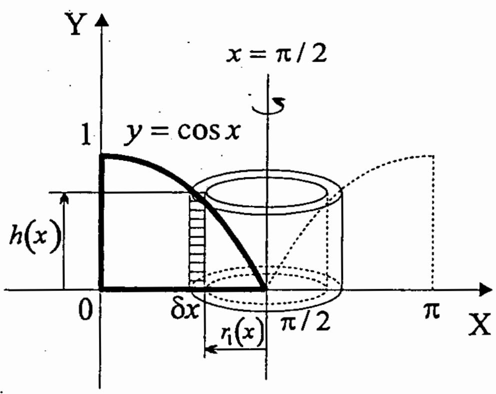
$$\therefore\quad V = \lim_{\delta x\to 0}\sum_{x=0}^{\pi/2} 2\pi x\sin x\,\delta x = 2\pi\int_0^{\pi/2} x\sin x\,dx$$

$$= -2\pi\int_0^{\pi/2} x\,d\cos x$$

$$= -2\pi\left(x\cos x\Big|_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\cos x\,dx\right) = 2\pi\sin x\Big|_0^{\pi/2} = 2\pi\,.$$

$\therefore$ the volume of the solid is 2π cubic units.

6 Solution



The typical cylindrical shell has radii $r_1(x) = \dfrac{\pi}{2} - x$, $r_2(x) = \dfrac{\pi}{2} - x + \delta x$, and height $h(x) = \cos x$.

This shell has volume

$$\delta V = \pi\left[\left(\frac{\pi}{2} - x + \delta x\right)^2 - \left(\frac{\pi}{2} - x\right)^2\right]h(x)$$

$$= 2\pi\left(\frac{\pi}{2} - x\right)\cos x\,\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore\quad V = \lim_{\delta x\to 0}\sum_{x=0}^{\pi/2} 2\pi\left(\frac{\pi}{2} - x\right)\cos x\,\delta x = 2\pi\int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)\cos x\,dx\,.$$

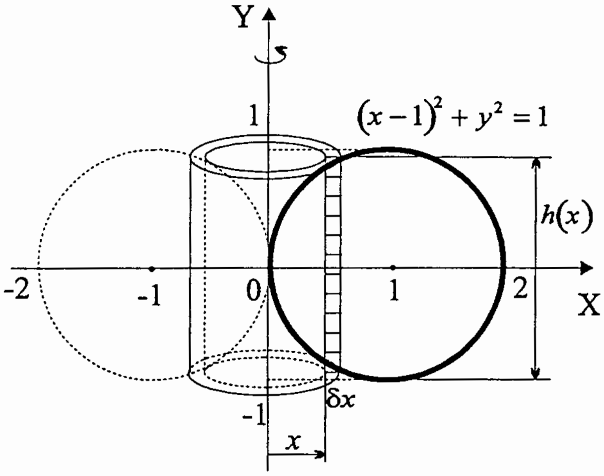
Substitution $x = \dfrac{\pi}{2} - x'$, $dx = -dx'$ gives

$$V = -2\pi\int_{\pi/2}^{0} x'\sin x'\,dx' = -2\pi\int_0^{\pi/2} x'\,d\cos x' = -2\pi\left(x'\cos x'\Big|_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\cos x'\,dx'\right)$$

$$= 2\pi\sin x'\Big|_0^{\pi/2} = 2\pi\,.$$

$\therefore$ the volume of the solid is 2π cubic units.

7 Solution



The typical cylindrical shell has radii x, $x+\delta x$. Height of the shell is obtained from

$$(x-1)^2+y^2=1$$

$$y^2=1-(x-1)^2$$

$$h(x)=2y=2\sqrt{1-(x-1)^2}\,.$$

The shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)$$

$$=4\pi x\sqrt{1-(x-1)^2}\,\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{2}4\pi x\sqrt{1-(x-1)^2}\,\delta x=4\pi\int_0^2 x\sqrt{1-(x-1)^2}\,dx\,.$$

Substitution $x=x'+1$, $dx=dx'$ gives

$$V=4\pi\int_{-1}^{1}(x'+1)\sqrt{1-x'^2}\,dx'=4\pi\int_{-1}^{1}x'\sqrt{1-x'^2}\,dx'+4\pi\int_{-1}^{1}\sqrt{1-x'^2}\,dx'\,.$$

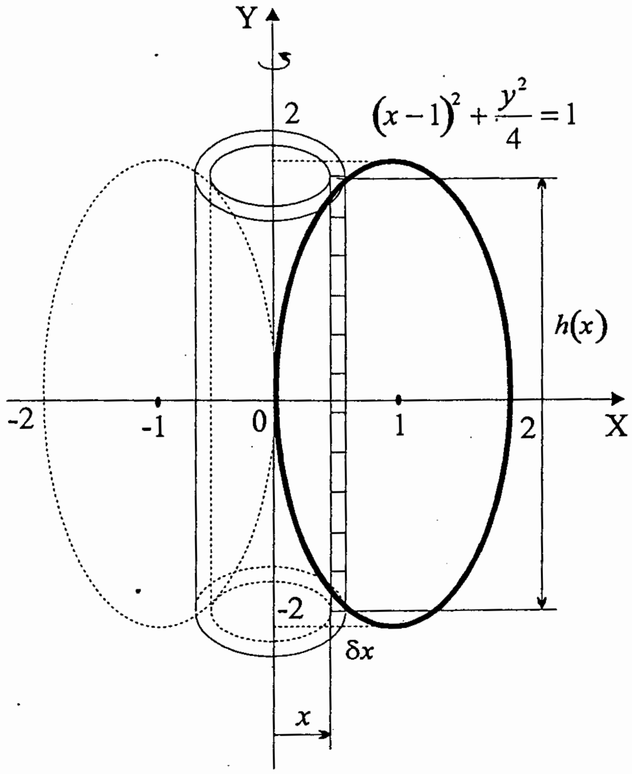
The first integral is equal to zero since the integrand is odd. Substitution $x'=\sin\varphi$, $dx'=\cos\varphi\,d\varphi$ into the second integral gives

$$V=4\pi\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\varphi}\,\cos\varphi\,d\varphi=4\pi\int_{-\pi/2}^{\pi/2}\cos^2\varphi\,d\varphi=4\pi\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\varphi}{2}\,d\varphi$$

$$=2\pi\left(\varphi+\frac{\sin 2\varphi}{2}\right)\Bigg|_{-\pi/2}^{\pi/2}=2\pi^2\,.$$

$\therefore$ the volume of the solid is $2\pi^2$ cubic units.

8 Solution



The typical cylindrical shell has radii x, $x+\delta x$. Height of the shell is obtained from

$$(x-1)^2+\frac{y^2}{4}=1$$

$$y^2=4\left[1-(x-1)^2\right]$$

$$h(x)=2y=4\sqrt{1-(x-1)^2}\,.$$

The shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)$$

$$=8\pi x\sqrt{1-(x-1)^2}\,\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{2}8\pi x\sqrt{1-(x-1)^2}\,\delta x$$

$$=8\pi\int_0^2 x\sqrt{1-(x-1)^2}\,dx\,.$$

Substitution $x=x'+1$, $dx=dx'$ gives

$$V=8\pi\int_{-1}^{1}(x'+1)\sqrt{1-x'^2}\,dx'=8\pi\int_{-1}^{1}x'\sqrt{1-x'^2}\,dx'+8\pi\int_{-1}^{1}\sqrt{1-x'^2}\,dx'.$$

The first integral is equal to zero since the integrand is odd. Substitution $x'=\sin\varphi$, $dx'=\cos\varphi\,d\varphi$ into the second integral gives

$$V=8\pi\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\varphi}\,\cos\varphi\,d\varphi=8\pi\int_{-\pi/2}^{\pi/2}\cos^2\varphi\,d\varphi=8\pi\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\varphi}{2}\,d\varphi$$

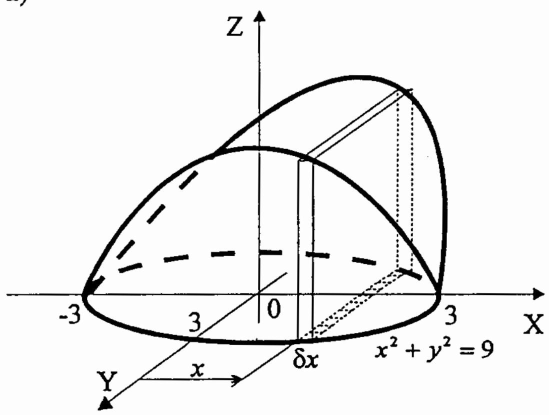
$$=4\pi\left(\varphi+\frac{\sin 2\varphi}{2}\right)\Bigg|_{-\pi/2}^{\pi/2}=4\pi^2\,.$$

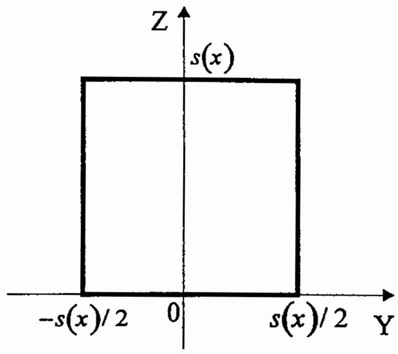
$\therefore$ the volume of the solid is $4\pi^2$ cubic units.

Exercise 6.3

1 Solution

a)





The slice is a square with area of cross-section A, thickness δx.

$$A(x) = s^2(x)$$

$$s(x) = 2\sqrt{9 - x^2}$$

$$\therefore \quad A(x) = 4(9 - x^2).$$

The slice has volume

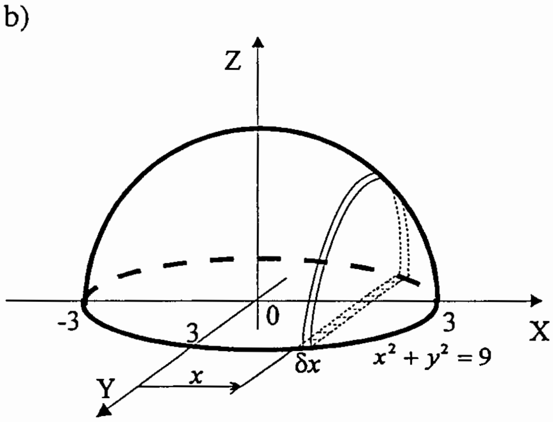
$$\delta V = A(x)\delta x = 4(9 - x^2)\delta x.$$

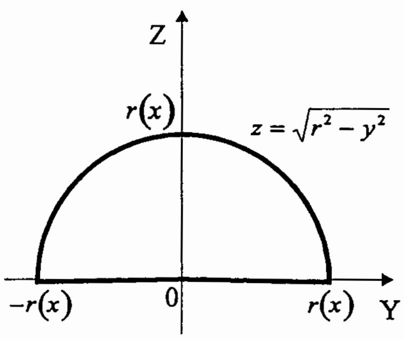
Then the volume of the solid is

$$V = \lim_{\delta x \to 0} \sum_{x=-3}^{3} 4(9 - x^2)\delta x = 4\int_{-3}^{3} (9 - x^2)\,dx = 4\left(9x - \frac{x^3}{3}\right)\Bigg|_{-3}^{3} = 144.$$

$\therefore$ the volume of the solid is 144 cubic units.

b)





The slice is a semicircle with area of cross-section A, thickness δx.

$$A(x) = \frac{\pi r^2(x)}{2}$$

$$r(x) = \sqrt{9 - x^2}$$

$$\therefore \quad A(x) = \frac{\pi(9-x^2)}{2}.$$

The slice has volume

$$\delta V = A(x)\delta x = \frac{\pi(9-x^2)}{2}\delta x.$$

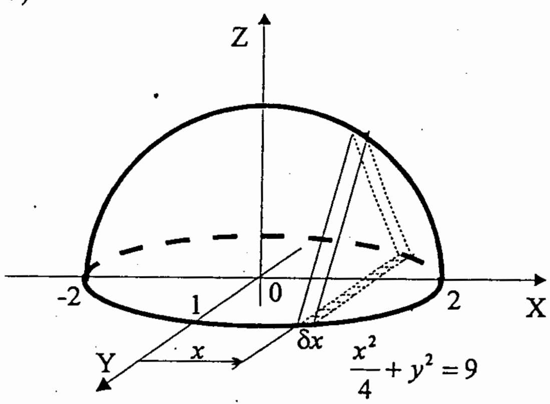
Then the volume of the solid is

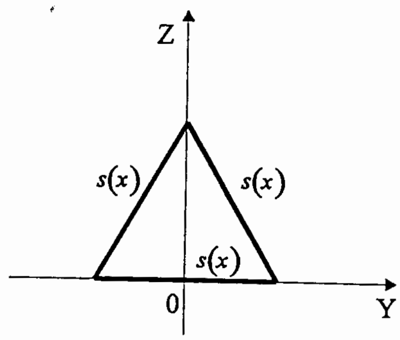
$$V = \lim_{\delta x \to 0} \sum_{x=-3}^{3} \frac{\pi(9-x^2)}{2}\delta x = \frac{\pi}{2}\int_{-3}^{3}(9-x^2)dx = \frac{\pi}{2}\left(9x - \frac{x^3}{3}\right)\Bigg|_{-3}^{3} = 18\pi.$$

$\therefore$ the volume of the solid is 18π cubic units.

2 Solution

a)





The slice is an equilateral triangle with area of cross-section A, thickness δx.

$$A(x) = \frac{\sqrt{3}\,s^2(x)}{4}$$

$$r(x) = 2\sqrt{1 - \frac{x^2}{4}}$$

$$\therefore \quad A(x) = \sqrt{3}\left(1 - \frac{x^2}{4}\right).$$

The slice has volume

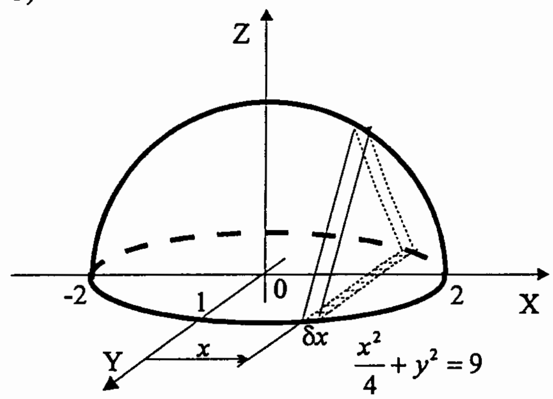
$$\delta V = A(x)\delta x = \sqrt{3}\left(1 - \frac{x^2}{4}\right)\delta x.$$

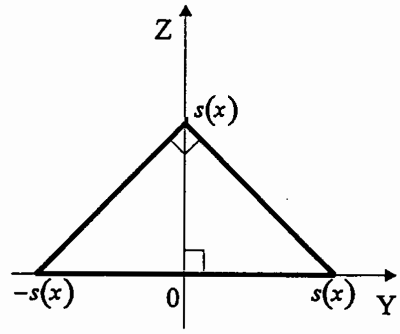
Then the volume of the solid is

$$V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} \sqrt{3}\left(1 - \frac{x^2}{4}\right)\delta x = \sqrt{3}\int_{-2}^{2}\left(1 - \frac{x^2}{4}\right)dx = \sqrt{3}\left(x - \frac{x^3}{4\cdot 3}\right)\Bigg|_{-2}^{2} = \frac{8}{\sqrt{3}}.$$

$\therefore$ the volume of the solid is $\frac{8}{\sqrt{3}}$ cubic units.

b)





The slice is an isosceles right-angled triangle with area of cross-section A, thickness δx.

$$A(x)=s^2(x)$$

$$s(x)=\sqrt{1-\frac{x^2}{4}}$$

$$\therefore \quad A(x)=1-\frac{x^2}{4}.$$

The slice has volume

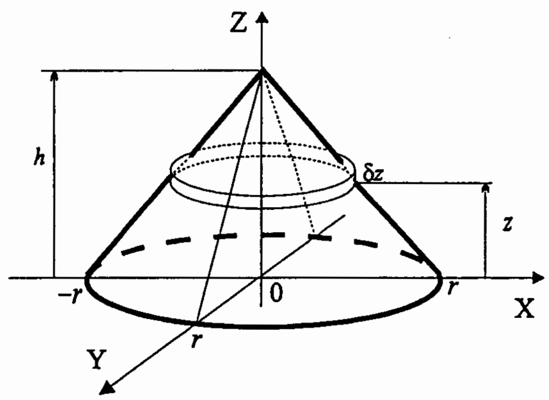
$$\delta V = A(x)\delta x = \left(1-\frac{x^2}{4}\right)\delta x.$$

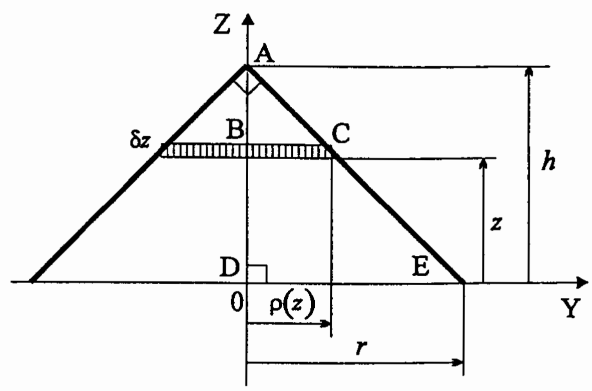
Then the volume of the solid is

$$V=\lim_{\delta x\to 0}\sum_{x=-2}^{2}\left(1-\frac{x^2}{4}\right)\delta x=\int_{-2}^{2}\left(1-\frac{x^2}{4}\right)dx=\left(x-\frac{x^3}{4\cdot 3}\right)\Bigg|_{-2}^{2}=\frac{8}{3}.$$

$\therefore$ the volume of the solid is $\frac{8}{3}$ cubic units.

3 Solution





Slicing the cone parallel to its base gives circular slices of radius ρ, thickness δz, and z is the height of the slice above the base.

$$\triangle ABC \;|||\; \triangle ADE \Rightarrow \frac{BC}{DE}=\frac{AB}{AD}\Rightarrow \frac{\rho}{r}=\frac{h-z}{h}\Rightarrow \rho=\frac{r(h-z)}{h}.$$

The slice has volume

$$\delta V = \pi\rho^2(z)\delta z = \pi\left(\frac{r}{h}\right)^2 (h-z)^2\,\delta z\,.$$

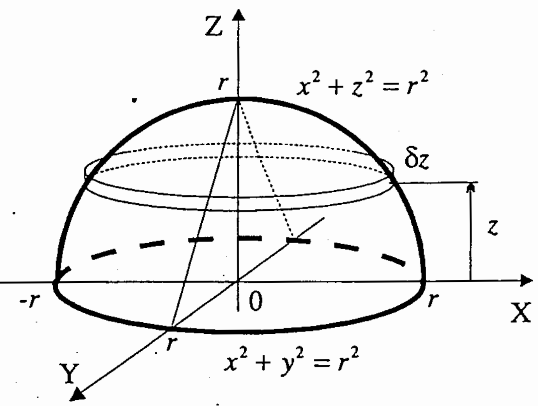
Then the volume of the solid is

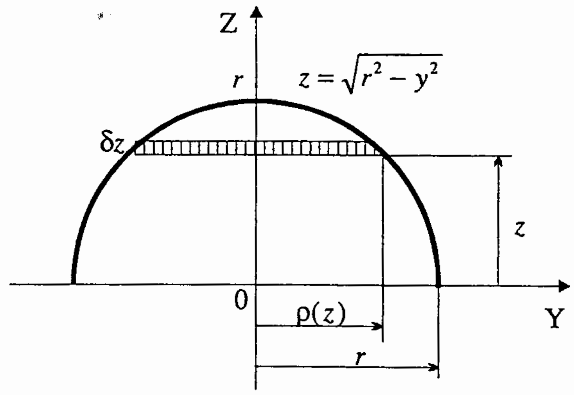
$$V = \lim_{\delta z\to 0}\sum_{z=0}^{h}\pi\left(\frac{r}{h}\right)^2 (h-z)^2\,\delta z = \pi\left(\frac{r}{h}\right)^2 \int_0^h (h-z)^2\,dz\,.$$

Substitution $z = h - z'$, $dz = -dz'$ gives

$$V = -\pi\left(\frac{r}{h}\right)^2 \int_h^0 z'^2\,dz' = -\pi\left(\frac{r}{h}\right)^2 \left.\frac{z'^3}{3}\right|_h^0 = \pi\frac{hr^2}{3}\,.$$

4 Solution





Slicing the hemisphere parallel to its base $x^2 + y^2 = r^2$ gives circular slices of radius ρ, thickness δz, and z is the height of the slice above the base. The area of cross-section of the slice is

$$A(z) = \pi\rho^2(z)$$

$$\rho(z) = \sqrt{r^2 - z^2}$$

$$\therefore\quad A(z) = \pi\left(r^2 - z^2\right).$$

The slice has volume

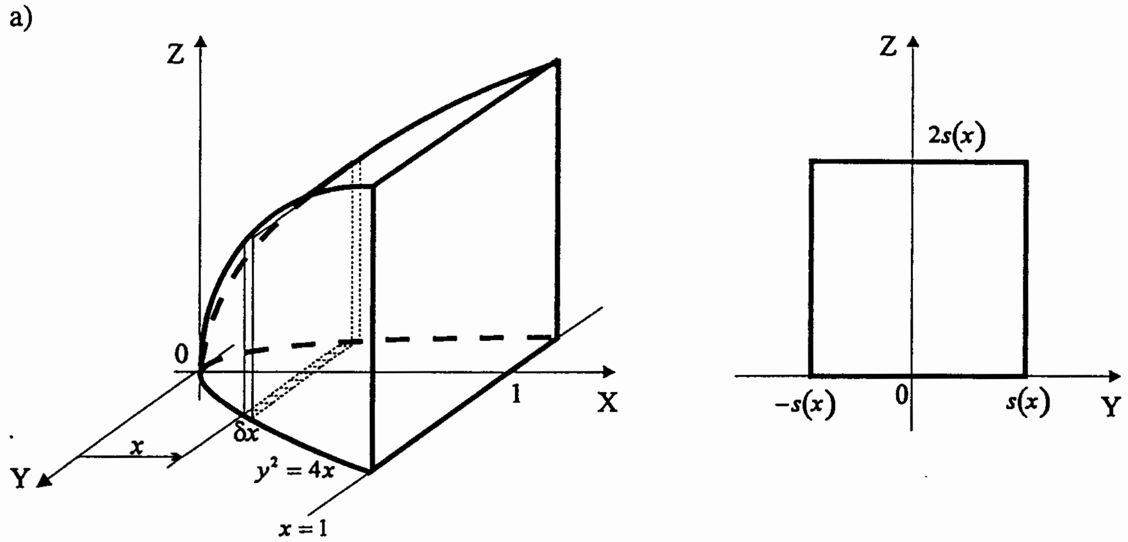
$$\delta V = A(x)\delta z = \pi\left(r^2 - z^2\right)\delta z\,.$$

Then the volume of the solid is

$$V = \lim_{\delta z\to 0}\sum_{z=0}^{r}\pi\left(r^2 - z^2\right)\delta z = \pi\int_0^r\left(r^2 - z^2\right)dz = \pi\left.\left(r^2 z - \frac{z^3}{3}\right)\right|_0^r = \frac{2\pi r^3}{3}\,.$$

5 Solution

a)



The latus rectum of the parabola $y^2 = 4x$ is the line $x = 1$. The slice is a square with area of cross-section A, thickness δx.

$$A(x) = (2s(x))^2$$
$$s(x) = 2\sqrt{x}$$
$$\therefore \quad A(x) = 16x.$$

The slice has volume

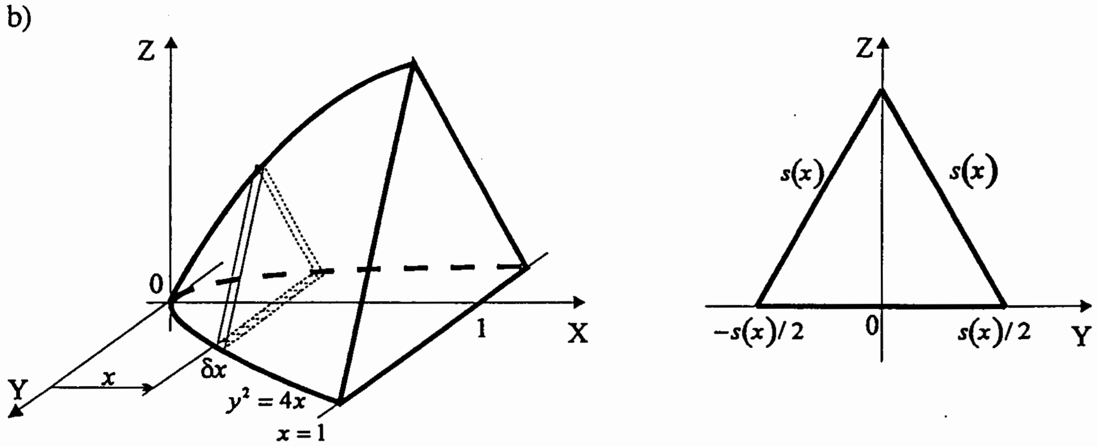
$$\delta V = A(x)\delta x = 16x\,\delta x.$$

Then the volume of the solid is

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 16x\,\delta x = 16\int_0^1 x\,dx = 16\frac{x^2}{2}\Big|_0^1 = 8.$$

$\therefore$ the volume of the solid is 8 cubic units.

b)



The latus rectum of the parabola $y^2 = 4x$ is the line $x = 1$. The slice is an equilateral triangle with area of cross-section A, thickness δx.

$$A(x)=\frac{\sqrt{3}\,s^2(x)}{4}$$

$$s(x)=2\sqrt{x}$$

$$\therefore\quad A(x)=4\sqrt{3}\,x\,.$$

The slice has volume

$$\delta V = A(x)\delta x = 4\sqrt{3}\,x\,\delta x\,.$$

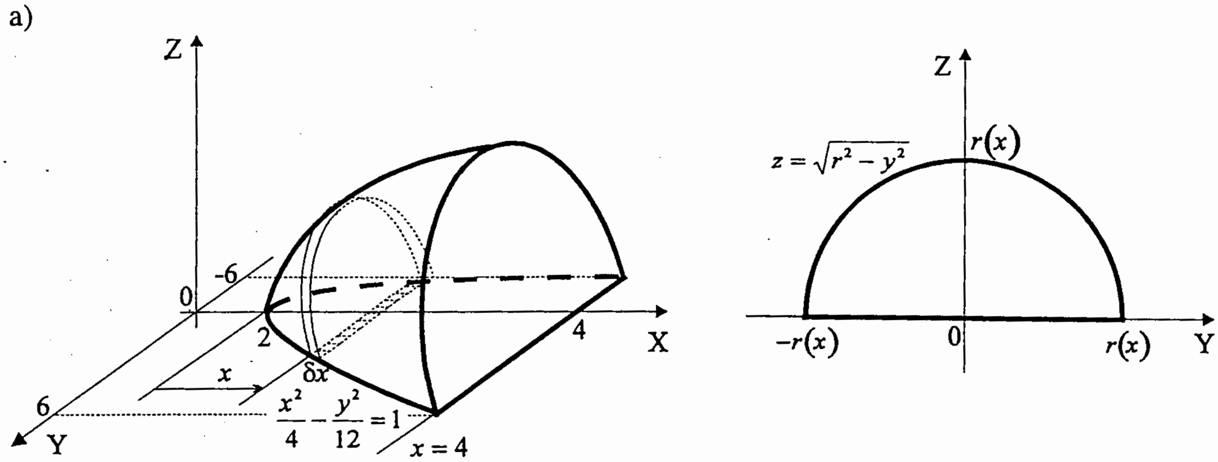
Then the volume of the solid is

$$V=\lim_{\delta x\to 0}\sum_{x=0}^{1}4\sqrt{3}\,x\,\delta x=4\sqrt{3}\int_0^1 x\,dx=4\sqrt{3}\left.\frac{x^2}{2}\right|_0^1=2\sqrt{3}\,.$$

$\therefore$ the volume of the solid is $2\sqrt{3}$ cubic units.

6 Solution

a)



The latus rectum of the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$ is the line $x=4$. The slice is a semicircle with radius r, area of cross-section A and thickness δx.

$$A(x)=\frac{\pi r^2(x)}{2}$$

$$r(x)=\sqrt{12}\cdot\sqrt{\frac{x^2}{4}-1}$$

$$\therefore\quad A(x)=6\pi\left(\frac{x^2}{4}-1\right).$$

The slice has volume

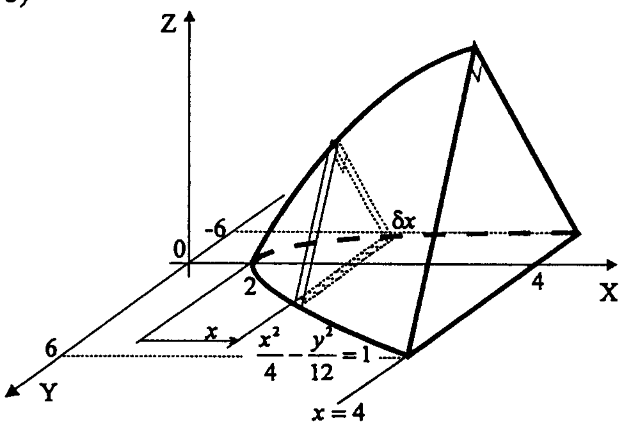
$$\delta V = A(x)\delta x = 6\pi\left(\frac{x^2}{4}-1\right)\delta x\,.$$

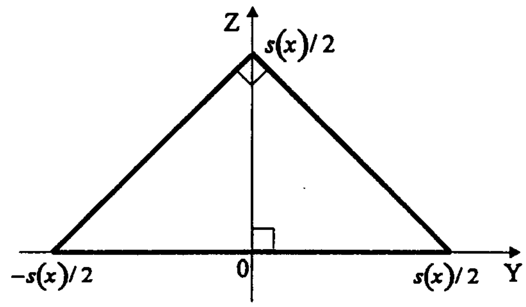
Then the volume of the solid is

$$V=\lim_{\delta x\to 0}\sum_{x=2}^{4}6\pi\left(\frac{x^2}{4}-1\right)\delta x=6\pi\int_2^4\left(\frac{x^2}{4}-1\right)dx=6\pi\left.\left(\frac{x^3}{4\cdot 3}-x\right)\right|_2^4=16\pi\,.$$

$\therefore$ the volume of the solid is 16π cubic units.

b)





The latus rectum of the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$ is the line $x=4$. The slice is an isosceles right-angled triangle with area of cross-section A, and thickness δx.

$$A(x)=\left(\frac{s(x)}{2}\right)^2$$

$$s(x)=2\cdot\sqrt{12}\sqrt{\frac{x^2}{4}-1}$$

$$\therefore\quad A(x)=12\left(\frac{x^2}{4}-1\right).$$

The slice has volume

$$\delta V=A(x)\delta x=12\left(\frac{x^2}{4}-1\right)\delta x.$$

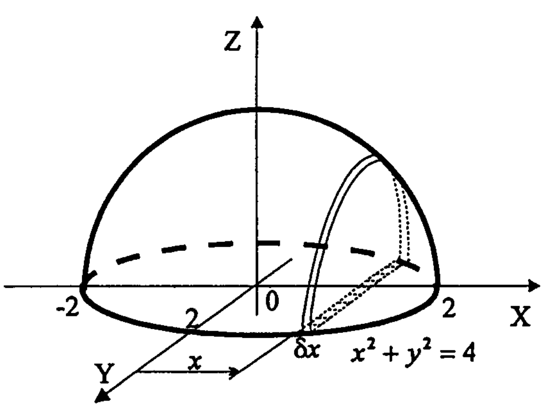
Then the volume of the slice is

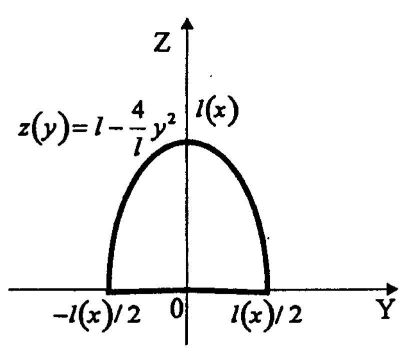
$$\delta V=A(x)\delta x=12\left(\frac{x^2}{4}-1\right)\delta x.$$

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=2}^{4}12\left(\frac{x^2}{4}-1\right)\delta x=12\int_2^4\left(\frac{x^2}{4}-1\right)dx=12\left(\frac{x^3}{4\cdot 3}-x\right)\Bigg|_2^4=32.$$

$\therefore$ the volume of the solid is 32 cubic units.

7 Solution





The slice is a parabolic segment with area of cross-section A, thickness δx. To calculate the area of the cross-section we need to deduce the equation $z(y)$ of the bounding parabola. We have

$$z(y) = \alpha + \beta\, y^2$$

$$z(y)\Big|_{y=0} = l \quad \Rightarrow \alpha = l$$

$$z(y)\Big|_{y=\pm\frac{l}{2}} = 0 \quad \Rightarrow l + \beta\left(\frac{l}{z}\right)^2 = 0 \quad \Rightarrow \quad \beta = -\frac{4}{l}$$

$$\therefore \quad z(y) = l - \frac{4}{l}y^2 .$$

The area of the segment is

$$A = \int_{-l/2}^{l/2} z(y)dy = \int_{-l/2}^{l/2}\left(l - \frac{4}{l}y^2\right)dy .$$

Integrand $l - \dfrac{4}{l}y^2$ is even

$$\therefore \; A = 2\int_0^{l/2}\left(l - \frac{4}{l}y^2\right)dy = 2\left(ly - \frac{4}{l}\frac{y^3}{3}\right)\Bigg|_0^{l/2} = \frac{2l^2}{3} .$$

Then, $l(x) = 2\sqrt{4-x^2}$,

$$A(x) = \frac{2}{3}\cdot\left(2\sqrt{4-x^2}\right)^2 = \frac{8\left(4-x^2\right)}{3} .$$

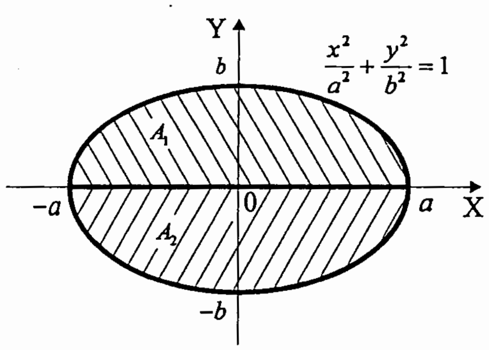
The volume of the solid is

$$\delta V = A(x)\delta x = \frac{8\left(4-x^2\right)}{3}\delta x .$$

$$\therefore \quad V = \lim_{\delta x\to 0}\sum_{x=-2}^{2}\frac{8\left(4-x^2\right)}{3}\delta x = \frac{8}{3}\int_{-2}^{2}\left(4-x^2\right)dx = \frac{8}{3}\left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} = \frac{256}{9} .$$

$\therefore$ the volume of the solid is $\dfrac{256}{9}$ cubic units.

8 Solution



a) Let the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is the sum of areas A_1 and A_2, i.e. $A = A_1 + A_2$, where $A_1 = A_2$. The area A_1 is enclosed by the curve $y(x) = b\sqrt{1 - \dfrac{x^2}{a^2}}$ and the x-axis. Hence $A_1 = \displaystyle\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx$.

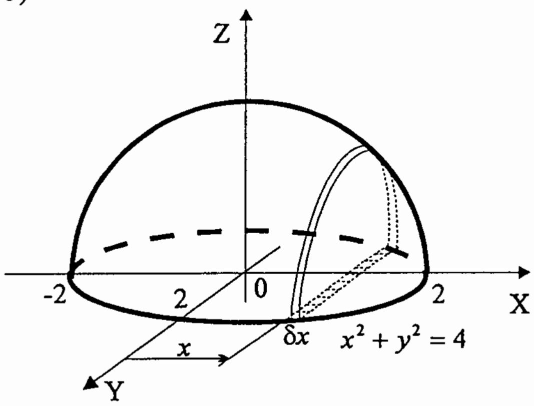
Substitution $x = a\sin\varphi$, $dx = a\cos\varphi\, d\varphi$ gives

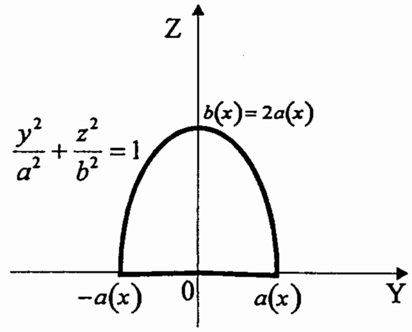
$$A_1 = ab\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\varphi}\cos\varphi\, d\varphi = ab\int_{-\pi/2}^{\pi/2}\cos^2\varphi\, d\varphi = ab\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\varphi}{2}d\varphi$$

$$= \frac{ab}{2}\left(\varphi + \frac{\sin 2\varphi}{2}\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{\pi ab}{2}.$$

$\therefore \quad A = 2A_1 = \pi ab.$

b)





The slice is a semi-ellipse with semi-minor axis a, semi-major axis b, area of cross-section A, thickness δx.

$$A = \frac{\pi ab}{2}$$

$$b = 2a$$

$$\therefore \quad A = \pi a^2.$$

$$a(x) = \sqrt{4-x^2}$$

$$\therefore \quad A(x) = \pi(4-x^2).$$

The volume of the slice is $\delta V = A(x)\delta x = \pi(4-x^2)\delta x$.

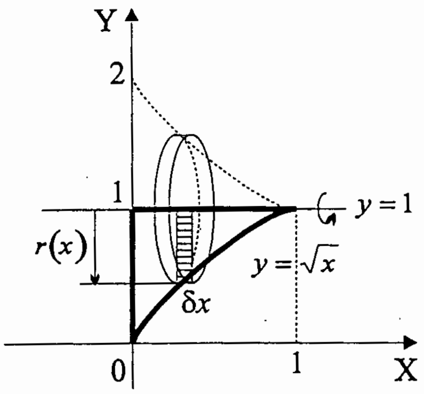
Then

$$\therefore \quad V = \lim_{\delta x\to 0}\sum_{x=-2}^{2}\pi(4-x^2)\delta x = \pi\int_{-2}^{2}(4-x^2)dx = \pi\left(4x-\frac{x^3}{3}\right)\Bigg|_{-2}^{2} = \frac{32\pi}{3}.$$

$\therefore$ the volume of the solid is $\frac{32\pi}{3}$ cubic units.

Diagnostic Test 6

1 Solution



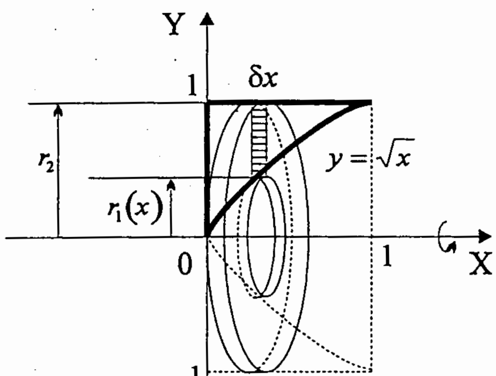
a) A slice taken perpendicular to the axis of rotation is a disk of thickness δx and radius $r(x) = 1 - \sqrt{x}$. The slice has volume

$$\delta V = \pi\left(1 - \sqrt{x}\right)^2 \delta x.$$

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \pi\left(1 - \sqrt{x}\right)^2 \delta x = \int_0^1 \pi\left(1 - \sqrt{x}\right)^2 dx$$

$$= \pi \int_0^1 \left(1 - 2\sqrt{x} + x\right) dx = \pi\left(x - 2 \cdot \frac{x^{3/2}}{3/2} + \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{\pi}{6}.$$

$\therefore$ the volume of the solid is $\frac{\pi}{6}$ cubic units.



b) A slice taken perpendicular to the axis of rotation is an annulus of thickness δx with radii $r_1(x) = \sqrt{x}$ and $r_2 = 1$. The slice has volume

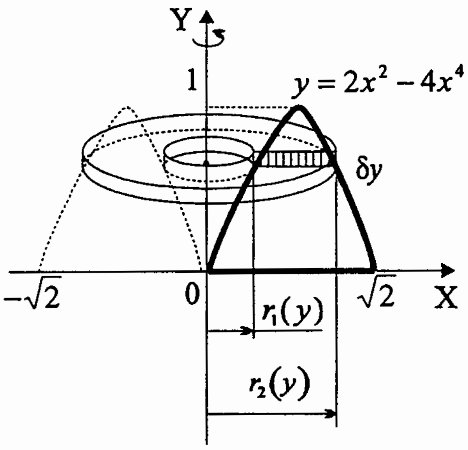
$$\delta V = \pi\left(r_2^2 - r_1^2\right)\delta x = \pi(1 - x)\delta x.$$

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{y=0}^{1} \pi(1 - x)\delta x = \int_0^1 \pi(1 - x)dx$$

$$= \pi\left(x - \frac{x^2}{2}\right)\Bigg|_0^1 = \frac{\pi}{2}.$$

$\therefore$ the volume of the solid is $\frac{\pi}{2}$ cubic units.

2 Solution



A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y) > r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of $y = 2r^2 - r^4$ considered as a biquadratic equation. The slice has volume $\delta V = \pi(r_2^2 - r_1^2)\delta y$.

$$y = 2r^2 - r^4$$
$$r^4 - 2r^2 + y = 0$$
$$z = r^2$$
$$z^2 - 2z + y = 0$$

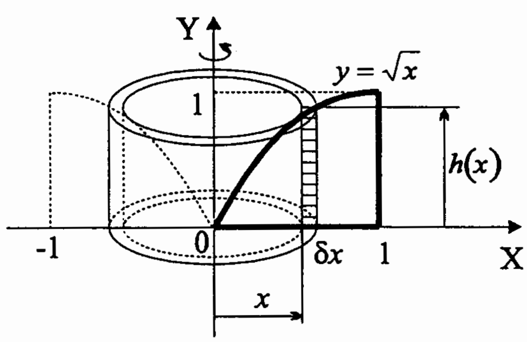
$$z_{1,2} = 1 \mp \sqrt{1-y}$$
$$r_1 = \sqrt{z_1} = \sqrt{1-\sqrt{1-y}}$$
$$r_2 = \sqrt{z_2} = \sqrt{1+\sqrt{1-y}}\ .$$
$$\therefore \quad \delta V = \pi(r_2^2 - r_1^2)\delta y = 2\pi\sqrt{1-y}\,\delta y\,.$$
$$\therefore \quad V = \lim_{\delta y \to 0} \sum_{y=0}^{1} 2\pi\sqrt{1-y}\,\delta y = \int_0^1 2\pi\sqrt{1-y}\,dy\,.$$

Substitution $y = 1 - y'$, $dy = -dy'$ gives

$$V = -2\pi\int_1^0 \sqrt{y'}\,dy' = -2\pi\frac{y'^{3/2}}{3/2}\Bigg|_1^0 = \frac{4\pi}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{4\pi}{3}$ cubic units.

3 Solution



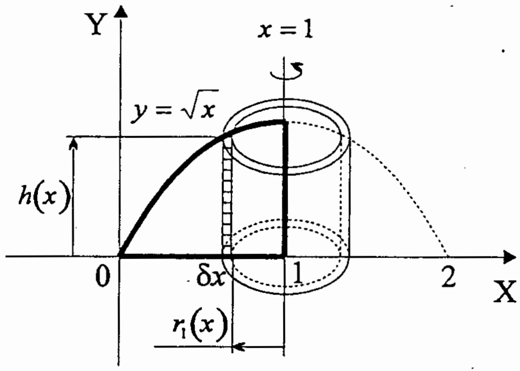
a) The typical cylindrical shell has radii x, $x + \delta x$, and height $h(x) = \sqrt{x}$. This shell has volume

$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]h(x) = 2\pi x^{3/2}\,\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x^{3/2}\,\delta x = 2\pi\int_0^1 x^{3/2}\,dx$$
$$= 2\pi\frac{x^{5/2}}{5/2}\Bigg|_0^1 = \frac{4\pi}{5}$$

$\therefore$ the volume of the solid is $\dfrac{4\pi}{5}$ cubic units.



b) The typical cylindrical shell has radii $r_1(x)=1-x$, $r_2(x)=1-x+\delta x$, and height $h(x)=\sqrt{x}$. This shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)=2\pi(1-x)\sqrt{x}\,\delta x$$

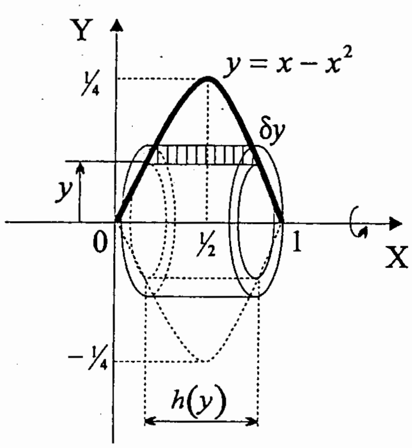
(ignoring $(\delta x)^2$).

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{1}2\pi(1-x)\sqrt{x}\,\delta x$$

$$=2\pi\int_0^1(1-x)\sqrt{x}\,dx=2\pi\left(\frac{x^{3/2}}{3/2}-\frac{x^{5/2}}{5/2}\right)\Bigg|_0^1=\frac{8\pi}{15}.$$

$\therefore$ the volume of the solid is $\dfrac{8\pi}{15}$ cubic units.

4 Solution



The typical cylindrical shell has radii y, $y+\delta y$, and height $h(y)$. We have

$$y=x-x^2$$
$$x^2-x+y=0$$
$$x_{1,2}=\frac{1\mp\sqrt{1-4y}}{2}.$$
$$\therefore\quad h(y)=x_2-x_1=\sqrt{1-4y}.$$

This shell has volume

$$\delta V=\pi\left[(4-y+\delta y)^2-(4-y)^2\right]h(y)$$
$$=2\pi y\sqrt{1-4y}\,\delta y\ \text{(ignoring } (\delta y)^2\text{)}.$$

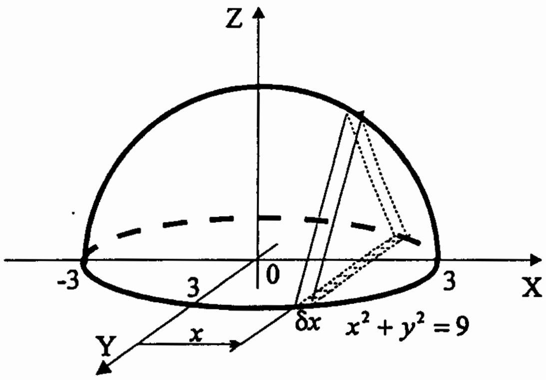
$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{1/4}2\pi y\sqrt{1-4y}\,\delta y=2\pi\int_0^{1/4}y\sqrt{1-4y}\,dy.$$

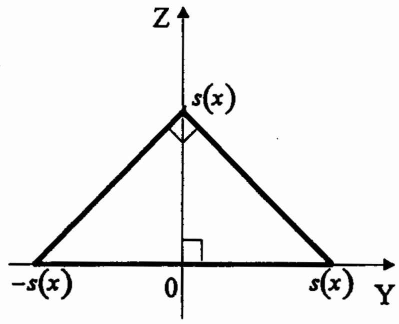
Substitution $y=\dfrac{1-y'}{4}$, $dy=-\dfrac{1}{4}dy'$ gives

$$V=-2\pi\cdot\frac{1}{16}\int_1^0(1-y')\sqrt{y'}\,dy'=-\frac{\pi}{8}\left(\frac{y'^{3/2}}{3/2}-\frac{y'^{5/2}}{5/2}\right)\Bigg|_1^0=\frac{\pi}{30}.$$

$\therefore$ the volume of the solid is $\dfrac{\pi}{30}$ cubic units.

5 Solution





The slice is a triangular segment with area of cross-section A, thickness δx.

$$A(x)=s^2(x)$$

$$s(x)=\sqrt{9-x^2}$$

$$\therefore\quad A(x)=9-x^2.$$

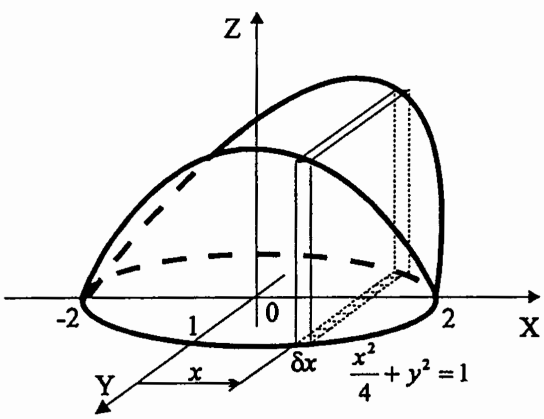
Hence the volume of the slice is $\delta V = A(x)\delta x = (9-x^2)\delta x$ .The volume of the solid is

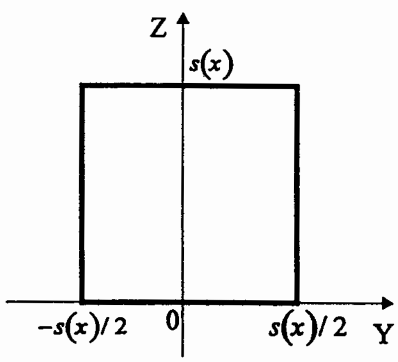
$$V=\lim_{\delta x\to 0}\sum_{x=-3}^{3}(9-x^2)\,\delta x=\int_{-3}^{3}(9-x^2)\,dx=2\int_{0}^{3}(9-x^2)\,dx$$

$$=2\left(9x-\frac{x^3}{3}\right)\Bigg|_0^3=36.$$

$\therefore$ the volume of the solid is 36 cubic units.

6 Solution





The slice is a square with area of cross-section A, thickness δx.

$$A(x)=s^2(x)$$

$$s(x)=\sqrt{4-x^2}$$

$$\therefore \quad A(x) = (4 - x^2).$$

The slice has volume

$$\delta V = A(x)\delta x = (4 - x^2)\delta x.$$

Then the volume of the solid is

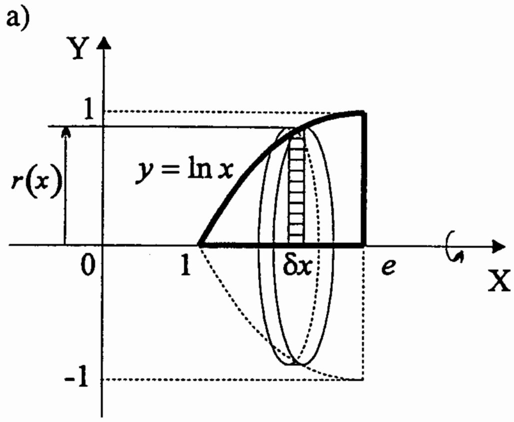
$$V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} (4 - x^2)\delta x = \int_{-2}^{2} (4 - x^2)\,dx = \left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} = \frac{32}{3}.$$

$\therefore$ the volume of the solid is $\frac{32}{3}$ cubic units.

Further Questions 6

1 Solution

a)



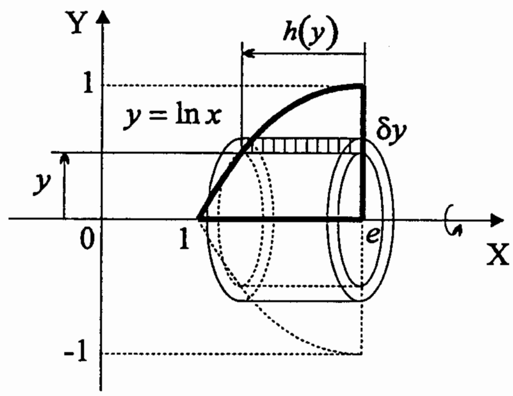
i) A slice taken perpendicular to the axis of rotation is a disk of thickness δx and radius $r(x) = \ln x$. The slice has volume

$$\delta V = \pi r^2(x)\delta x = \pi \ln^2 x\,\delta x.$$

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=1}^{e} \pi \ln^2 x\,\delta x = \int_1^e \pi \ln^2 x\,dx$$

$$= \pi x \ln^2 x\Big|_1^e - \pi\int_1^e \left(2\ln x \cdot \frac{1}{x}\right) x\,dx$$

$$= \pi e - 2\pi\int_1^e \ln x\,dx = \pi e - 2\pi\left(x\ln x\Big|_1^e - \int_1^e x\cdot\frac{1}{x}dx\right) = -\pi e + 2\pi x\Big|_1^e = \pi(e-2).$$



ii) The typical cylindrical shell has radii y, $y+\delta y$, and height $h(y)$ to be found. We have

$$y = \ln x$$
$$x = e - h(y)$$
$$y = \ln(e - h(y))$$
$$\therefore \quad h(y) = e - e^y.$$

The shell has volume

$$\delta V = \pi\left[(y+\delta y)^2 - y^2\right]h(y) = 2\pi\left(e - e^y\right)y\,\delta y$$

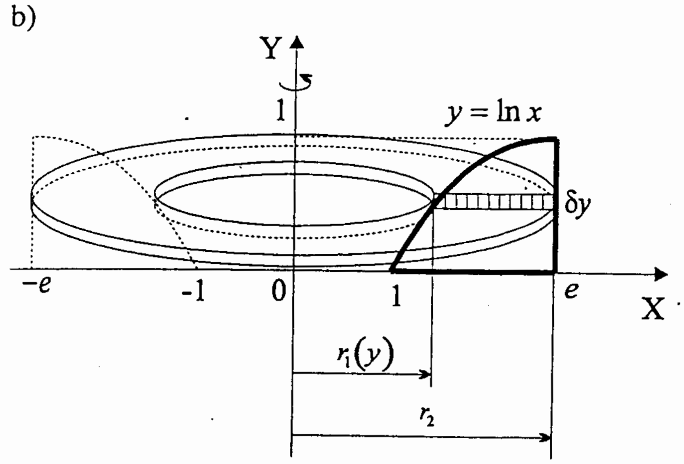
(ignoring $(\delta y)^2$).

$$\therefore \quad V = \lim_{\delta y \to 0}\sum_{y=0}^{1} 2\pi\left(e-e^y\right)y\,\delta y = 2\pi\int_0^1 \left(e - e^y\right)y\,dy$$

$$= 2\pi\left[\frac{ey^2}{2}\Big|_0^1 - \int_0^1 y\,de^y\right] = 2\pi\left[\frac{e}{2} - \left(ye^y - e^y\right)\Big|_0^1\right] = \pi(e-2).$$

$\therefore$ the volume of the solid is $\pi(e-2)$ cubic units.

b)



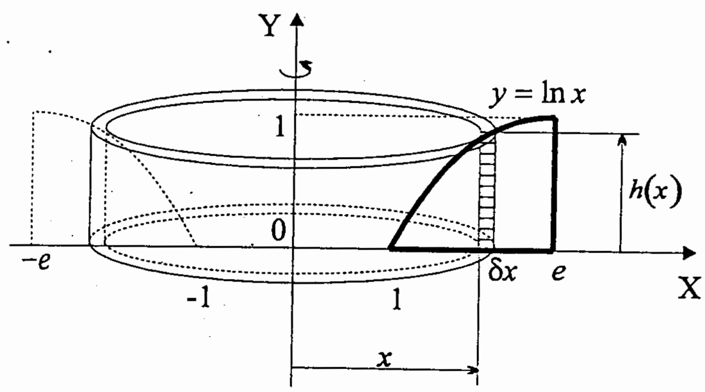
i) A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)=e^y$ and $r_2=e$. The slice has volume

$$\delta V=\pi\left(r_2^2-r_1^2\right)\delta y=\pi\left(e^2-e^{2y}\right)\delta y$$

$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{1}\pi\left(e^2-e^{2y}\right)\delta y$$

$$=\pi\int_0^1\left(e^2-e^{2y}\right)dy=\pi\left(e^2y-\frac{e^{2y}}{2}\right)\Bigg|_0^1$$

$$=\frac{\pi}{2}\left(e^2+1\right).$$



ii) The typical cylindrical shell has radii x, $x+\delta x$, and height

$$h(x)=\ln x.$$

This shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)$$

$$=2\pi x\ln x\,\delta x$$

(ignoring $(\delta y)^2$).

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=1}^{e}2\pi x\ln x\,\delta x$$

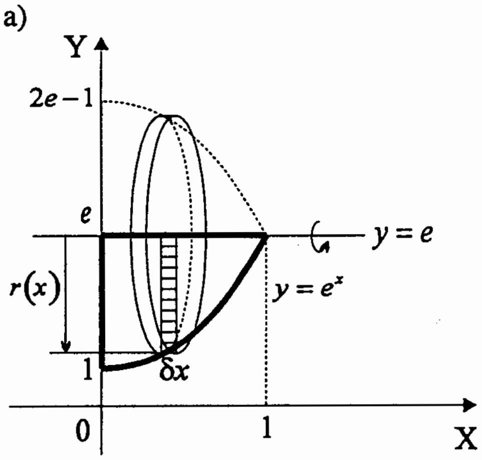
$$=2\pi\int_1^e x\ln x\,dx=2\pi\int_1^e\ln x\,d\frac{x^2}{2}=2\pi\left[\ln x\frac{x^2}{2}\Bigg|_1^e-\int_1^e\frac{1}{x}\cdot\frac{x^2}{2}dx\right]=2\pi\left[\frac{e^2}{2}-\frac{1}{2}\int_1^e x\,dx\right]$$

$$=\pi\left(e^2-\frac{x^2}{2}\Bigg|_1^e\right)=\frac{\pi}{2}\left(e^2+1\right).$$

$\therefore$ the volume of the solid is $\pi\left(e^2+1\right)$ cubic units.

2 Solution

a)



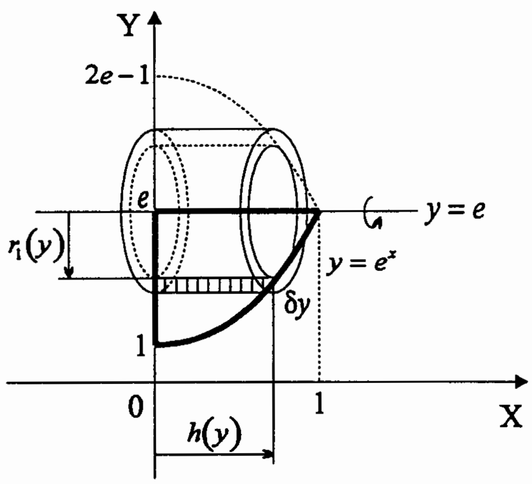
i) A slice taken perpendicular to the axis of rotation is a disk of thickness δx and radius $r(x)=e-e^x$. The slice has volume

$$\delta V = \pi r^2(x)\delta x = \pi\left(e-e^x\right)^2 \delta x.$$

$$\therefore \quad V = \lim_{\delta x\to 0}\sum_{x=0}^{1}\pi\left(e-e^x\right)^2\delta x = \int_0^1 \pi\left(e-e^x\right)^2 dx$$

$$= \pi\int_0^1\left(e^2-2e^{x+1}+e^{2x}\right)dx$$

$$= \pi\left(xe^2-2e^{x+1}+\frac{e^{2x}}{2}\right)\Bigg|_0^1 = \frac{\pi}{2}\left(-e^2+4e-1\right).$$



ii) The typical cylindrical shell has radii $r_1(y)=e-y$, $r_2(y)=e-y+\delta y$, and height $h(y)=\ln y$. This shell has volume

$$\delta V = \pi\left[\left(e-y+\delta y\right)^2-\left(e-y\right)^2\right]h(y)$$

$$= 2\pi(e-y)\ln y\,\delta y \text{ (ignoring } (\delta y)^2\text{).}$$

$$\therefore \quad V = \lim_{\delta y\to 0}\sum_{y=1}^{e}2\pi(e-y)\ln y\,\delta y$$

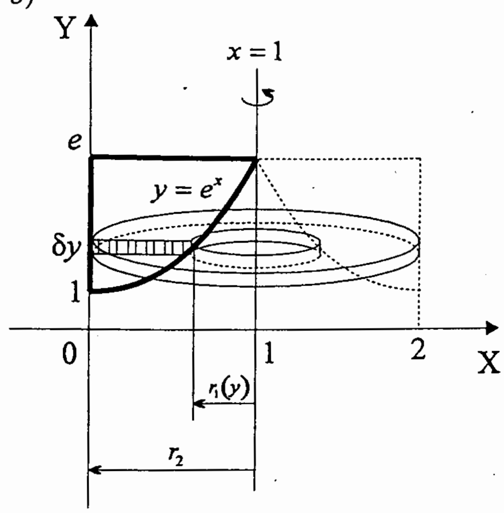
$$= 2\pi\int_1^e (e-y)\ln y\,dy = -2\pi\int_1^e \ln y\, d\frac{(e-y)^2}{2}$$

$$= -2\pi\left[\frac{(e-y)^2}{2}\ln y\Bigg|_1^e - \int_1^e \frac{(e-y)^2}{2}\cdot\frac{1}{y}dy\right] = \pi\int_1^e\left(\frac{e^2}{y}-2e+y\right)dy = \pi\left(e^2\ln y-2ey+\frac{y^2}{2}\right)\Bigg|_1^e$$

$$= \frac{\pi}{2}\left(-e^2+4e-1\right).$$

$\therefore$ the volume of the solid is $\frac{\pi}{2}\left(-e^2+4e-1\right)$ cubic units.

b)



i) A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)=1-\ln y$ and $r_2=1$. The slice has volume

$$\delta V=\pi\left(r_2^2-r_1^2\right)\delta y=\pi\left[1-(1-\ln y)^2\right]\delta y .$$

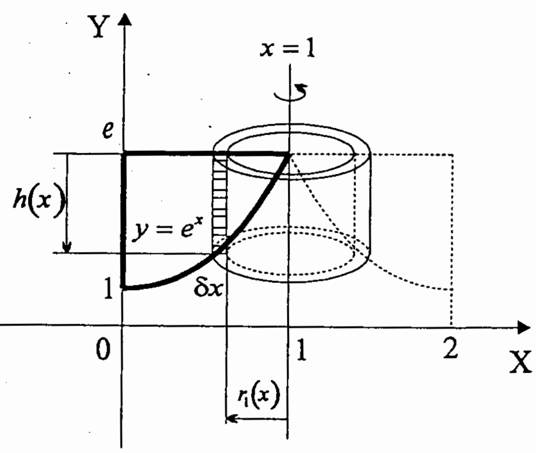
$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=1}^{e}\pi\left[1-(1-\ln y)^2\right]\delta y$$

$$=\pi\int_1^e\left[1-(1-\ln y)^2\right]dy=\pi\int_1^e\left(2\ln y-\ln^2 y\right)dy$$

$$=\pi\left[\left(2\ln y-\ln^2 y\right)y\Big|_1^e-\int_1^e\left(\frac{2}{y}-\frac{2\ln y}{y}\right)y\,dy\right]$$

$$=\pi\left[e-2\int_1^e(1-\ln y)dy\right]=\pi\left[e-2y\Big|_1^e+2\int_1^e\ln y\,dy\right]=\left[2-e+2\left(y\ln y\Big|_1^e-\int_1^e\frac{1}{y}\cdot y\,dy\right)\right]$$

$$=\pi\left[2+e-2\int_1^e dy\right]=\pi\left(2+e-2y\Big|_1^e\right)=\pi(4-e).$$



ii) The typical cylindrical shell has radii $r_1(x)=1-x$, $r_2(x)=1-x+\delta x$, and height $h(x)=e-e^x$. This shell has volume

$$\delta V=\pi\left[(1-x+\delta x)^2-(1-x)^2\right]h(x)$$

$$=2\pi(1-x)\left(e-e^x\right)\delta x$$

(ignoring $(\delta y)^2$).

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{1}2\pi(1-x)\left(e-e^x\right)\delta x$$

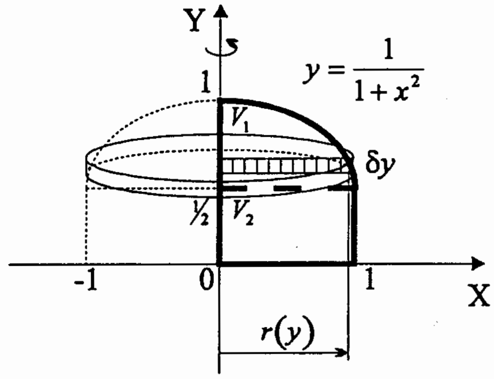
$$=\int_0^1 2\pi(1-x)\left(e-e^x\right)dx$$

$$=2\pi\left[e\int_0^1(1-x)dx-\int_0^1(1-x)e^x\,dx\right]=2\pi\left[e\left(x-\frac{x^2}{2}\right)\Bigg|_0^1-\int_0^1(1-x)de^x\right]$$

$$=2\pi\left[\frac{e}{2}-\left((1-x)e^x\Big|_0^1-\int_0^1(-1)\cdot e^x dx\right)\right]=2\pi\left[\frac{e}{2}+1-e^x\Big|_0^1\right]=\pi(4-e).$$

$\therefore$ the volume of the solid is $\pi(4-e)$ cubic units.

3 Solution



i) It is convenient to split volume V of the solid into volumes V_1 and V_2 (see figure).

1) volume V_1:

A slice taken perpendicular to the axis of rotation is a disk of thickness δy and radius $r(y)$. Deduce the equation of $r(y)$:

$$y=\frac{1}{1+r^2} \quad\Rightarrow\quad r=\sqrt{\frac{1}{y}-1}\,.$$

The slice has volume

$$\delta V_1=\pi r^2(y)\delta y=\pi\left(\frac{1}{y}-1\right)\delta y\,.$$

Hence

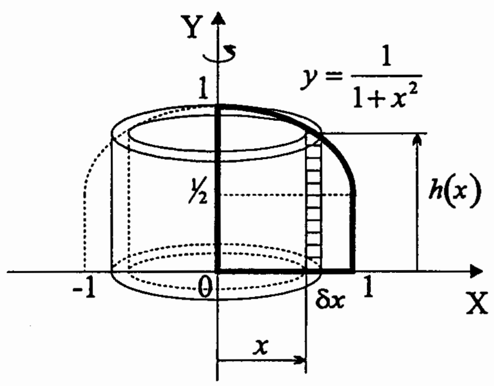
$$V_1=\lim_{\delta y\to 0}\sum_{y=\frac{1}{2}}^{1}\pi\left(\frac{1}{y}-1\right)\delta y=\pi\int_{\frac{1}{2}}^{1}\left(\frac{1}{y}-1\right)\delta y=\pi(\ln y-y)\Big|_{\frac{1}{2}}^{1}=\pi\left(\ln 2-\frac{1}{2}\right).$$

2) volume V_2:

This volume is a cylinder of radius $r=1$ and height $\frac{1}{2}$. Thus

$$V_2=\pi\cdot 1^2\cdot\frac{1}{2}=\frac{\pi}{2}\,.$$

$$\therefore\quad V=V_1+V_2=\pi\ln 2\,.$$



ii) The typical cylindrical shell has radii x, $x+\delta x$, and height $h(x)=\dfrac{1}{1+x^2}$. This shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)=\frac{2\pi x}{1+x^2}\delta x$$

(ignoring $(\delta x)^2$).

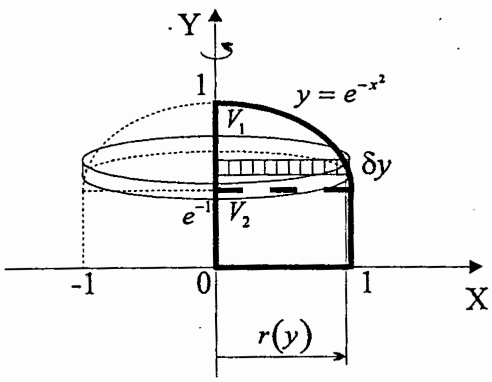
$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{1}\frac{2\pi x}{1+x^2}\delta x=\pi\int_0^1\frac{2x}{1+x^2}dx\,.$$

Substitution $z=1+x^2$, $dz=2x\,dx$ gives

$$V=\pi\int_1^2\frac{dz}{z}=\pi\ln z\Big|_1^2=\pi\ln 2\,.$$

$\therefore$ the volume of the solid is $\pi\ln 2$ cubic units.

4 Solution



i) It is convenient to split volume V of the solid into volumes V_1 and V_2 (see figure).

1) volume V_1:

A slice taken perpendicular to the axis of rotation is a disk of thickness δy and radius $r(y)$. Deduce the equation of $r(y)$:

$$y=e^{-r^2} \quad \Rightarrow \quad r=\sqrt{-\ln y}\,.$$

The slice has volume

$\delta V_1=\pi r^2(y)\delta y=-\pi\ln y\,\delta y$.

Hence

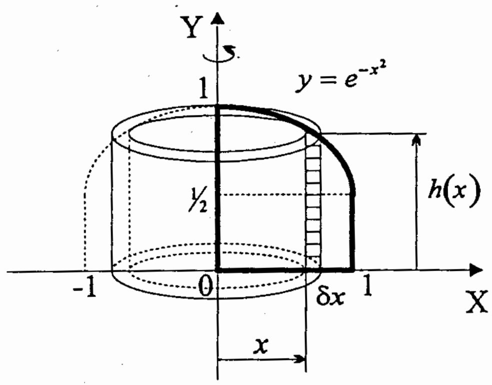
$$V_1=\lim_{\delta y\to 0}\sum_{y=e^{-1}}^{1}(-\pi\ln y)\delta y=-\pi\int_{e^{-1}}^{1}\ln y\,\delta y=-\pi(y\ln y-y)\Big|_{e^{-1}}^{1}=\pi\left(1-\frac{2}{e}\right).$$

2) volume V_2:

This volume is a cylinder of radius $r=1$ and height e^{-1}. Thus

$$V_2=\pi\cdot(1)^2\cdot\frac{1}{e}=\frac{\pi}{e}.$$

$$\therefore\quad V=V_1+V_2=\pi\left(1-e^{-1}\right).$$



ii) The typical cylindrical shell has radii x, $x+\delta x$, and height $h(x)=e^{-x^2}$. This shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)=2\pi x e^{-x^2}\delta x$$

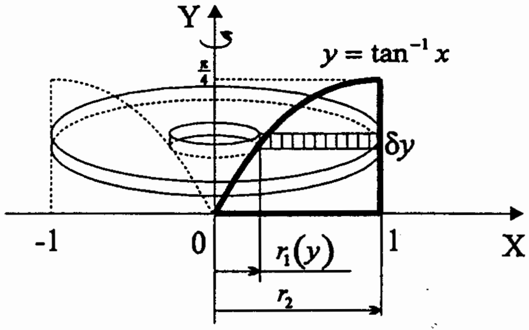
(ignoring $(\delta x)^2$).

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{1}2\pi x e^{-x^2}\,\delta x=2\pi\int_0^1 x e^{-x^2}\,dx$$

$$=\pi\int_0^1 e^{-x^2}\,dx^2=-\pi e^{-x^2}\Big|_0^1=\pi\left(1-e^{-1}\right).$$

$\therefore$ the volume of the solid is $\pi\left(1-e^{-1}\right)$ cubic units.

5 Solution



i) A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)=\tan y$ and $r_2=1$. The slice has volume

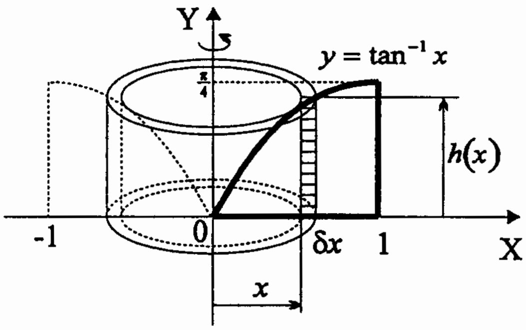
$$\delta V=\pi\left(r_2^2-r_1^2\right)\delta y=\pi\left[1-\tan^2 y\right]\delta y$$

$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=0}^{\pi/4}\pi\left(1-\tan^2 y\right)\delta y.$$

$$=\pi\int_0^{\pi/4}\left(1-\tan^2 y\right)dy$$

$$=\pi\int_0^{\pi/4}dy-\pi\int_0^{\pi/4}\frac{\sin^2 y}{\cos^2 y}dy=\pi y\Big|_0^{\pi/4}-\pi\int_0^{\pi/4}\sin y\, d\left(\frac{1}{\cos y}\right)$$

$$=\frac{\pi^2}{4}-\pi\left(\frac{\sin y}{\cos y}\Bigg|_0^{\pi/4}-\int_0^{\pi/4}\frac{1}{\cos y}d\sin y\right)=\frac{\pi^2}{4}-\pi+\pi\int_0^{\pi/4}dy=\frac{\pi}{2}(\pi-2).$$



ii) The typical cylindrical shell has radii x, $x+\delta x$, and height $h(x)=\tan^{-1}x$. This shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)=2\pi x\tan^{-1}x\,\delta x$$

(ignoring $(\delta x)^2$).

$$\therefore\quad V=\lim_{\delta x\to 0}\sum_{x=0}^{1}2\pi x\tan^{-1}x\,\delta x$$

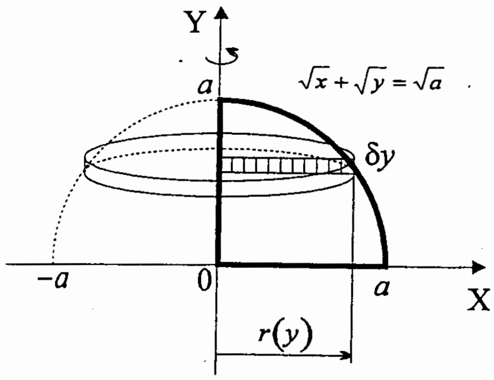
$$=2\pi\int_0^1 x\tan^{-1}x\,dx=\pi\int_0^1\tan^{-1}x\,dx^2$$

$$=\pi\left[x^2\tan^{-1}x\Big|_0^1-\int_0^1\frac{x^2}{1+x^2}dx\right]=\pi\left[\frac{\pi}{4}-\int_0^1\frac{\left(1+x^2\right)-1}{1+x^2}dx\right]=\pi\left[\frac{\pi}{4}-\int_0^1 dx+\int_0^1\frac{dx}{1+x^2}\right]$$

$$=\pi\left[\frac{\pi}{4}-x\Big|_0^1+\tan^{-1}x\Big|_0^1\right]=\frac{\pi}{2}(\pi-2).$$

$\therefore$ the volume of the solid is $\frac{\pi}{2}(\pi-2)$ cubic units.

6 Solution



i) A slice taken perpendicular to the axis of rotation is a disk of thickness δy and radius $r(y)$. Deduce the equation of $r(y)$:

$$\sqrt{r}+\sqrt{y}=\sqrt{a} \quad \Rightarrow \quad r=\left(\sqrt{a}-\sqrt{y}\right)^2.$$

The slice has volume

$$\delta V=\pi r^2(y)\delta y=\pi\left(\sqrt{a}-\sqrt{y}\right)^4\delta y.$$

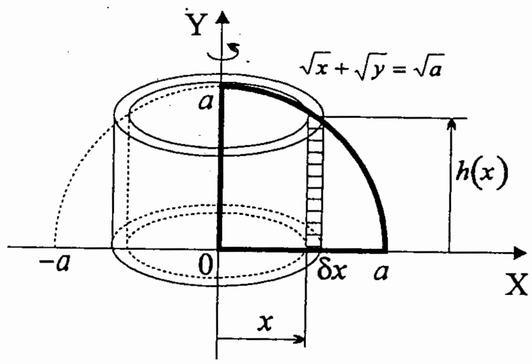
$$\therefore \quad V=\lim_{\delta y\to 0}\sum_{y=0}^{a}\pi\left(\sqrt{a}-\sqrt{y}\right)^4\delta y$$

$$=\pi\int_0^a\left(\sqrt{a}-\sqrt{y}\right)^4 dy.$$

Substitution $y=a(z+1)^2$, $dy=2a(z+1)dz$ yields

$$V=2\pi a\int_{-1}^{0}\left[\sqrt{a}-\sqrt{a}(z+1)\right]^4(z+1)dz=2\pi a^3\int_{-1}^{0}z^4(z+1)dz=2\pi a^3\left(\frac{z^6}{6}+\frac{z^5}{5}\right)\Bigg|_{-1}^{0}$$

$$=\frac{\pi a^3}{15}.$$



ii) The typical cylindrical shell has radii x, $x+\delta x$, and height $h(x)$.

$$\sqrt{x}+\sqrt{h}=\sqrt{a} \quad \Rightarrow$$

$$h(x)=\left(\sqrt{a}-\sqrt{x}\right)^2.$$

This shell has volume

$$\delta V=\pi\left[(x+\delta x)^2-x^2\right]h(x)$$

$$=2\pi x\left(\sqrt{a}-\sqrt{x}\right)^2\delta x \quad \text{(ignoring } (\delta x)^2\text{)}.$$

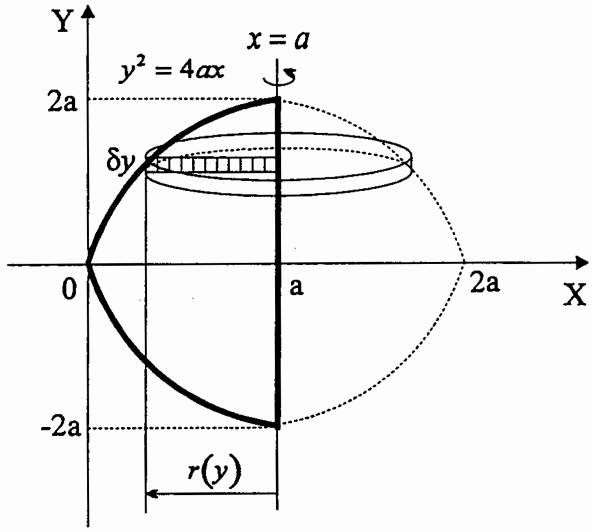
$$\therefore \quad V=\lim_{\delta x\to 0}\sum_{x=0}^{a}2\pi x\left(\sqrt{a}-\sqrt{x}\right)^2\delta x=2\pi\int_0^a x\left(\sqrt{a}-\sqrt{x}\right)^2 dx.$$

Substitution $x=az^2$, $dx=2az\,dz$ yields

$$V=4\pi a^3\int_0^1(1-z)^2z^3\,dz=4\pi a^3\int_0^1\left(1-2z+z^2\right)z^3\,dz=4\pi a^3\left(\frac{z^4}{4}-2\cdot\frac{z^5}{5}+\frac{z^6}{6}\right)\Bigg|_0^1=\frac{\pi a^3}{15}.$$

$\therefore$ the volume of the solid is $\dfrac{\pi a^3}{15}$ cubic units.

7 Solution



i) Latus rectum of the parabola $y^2 = 4ax$ is the line $x = a$. A slice taken perpendicular to the axis of rotation is a disk of thickness δy and radius $r(y) = a - \frac{y^2}{4a}$. The slice has volume

$$\delta V = \pi r^2(y)\delta y = \pi\left(a - \frac{y^2}{4a}\right)^2 \delta y.$$

Hence

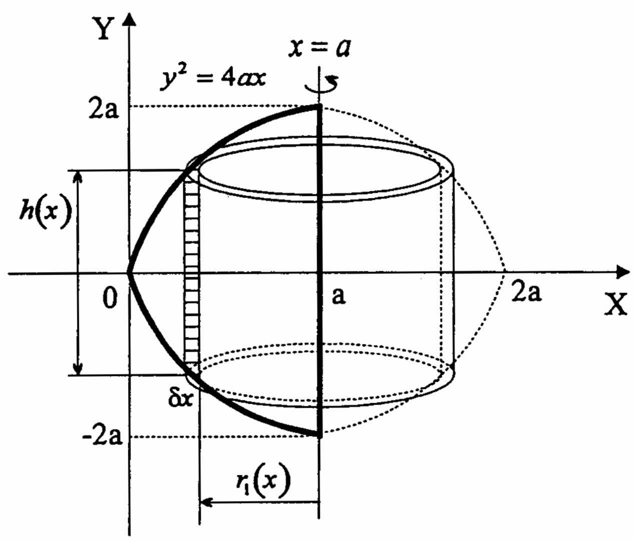
$$V = \lim_{\delta y \to 0} \sum_{y=-2a}^{2a} \pi\left(a - \frac{y^2}{4a}\right)^2 \delta y$$

$$= \pi \int_{-2a}^{2a} \left(a - \frac{y^2}{4a}\right)^2 dy.$$

Substitution $y = 2az$, $dy = 2a\,dz$ gives

$$V = 2\pi a^3 \int_{-1}^{1} \left(1 - z^2\right)^2 dz = 4\pi a^3 \int_{0}^{1} \left(1 - 2z^2 + z^4\right) dz$$

$$= 4\pi a^3 \left(z - 2\cdot\frac{z^3}{3} + \frac{z^5}{5}\right)\Bigg|_0^1 = \frac{32\pi a^3}{15}.$$



ii) Latus rectum of the parabola $y^2 = 4ax$ is the line $x = a$. The typical cylindrical shell has radii $r_1(x) = a - x$, $r_2(x) = a - x + \delta x$, and height $h(x) = 2\cdot\sqrt{4ax}$. This shell has volume

$$\delta V = \pi\left(r_2^2 - r_1^2\right)h(x)$$
$$= 8\pi(a - x)\sqrt{ax}\,\delta x$$

(ignoring $(\delta x)^2$).

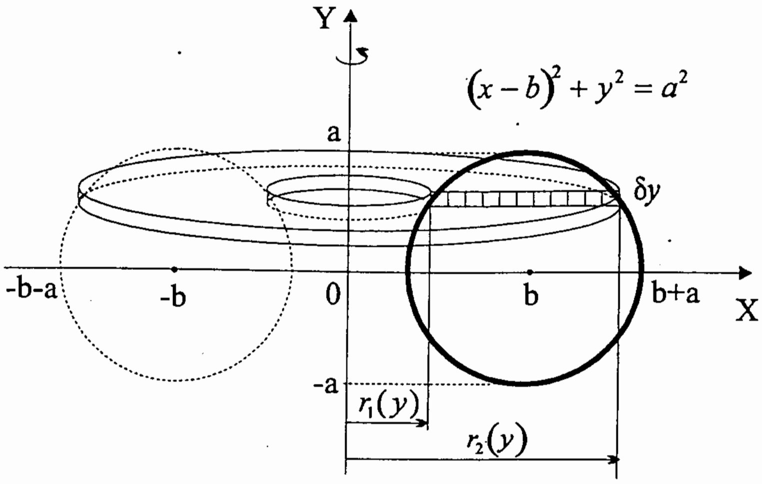
$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{a} 8\pi(a - x)\sqrt{ax}\,\delta x$$

$$= 8\pi\sqrt{a}\int_0^a (a - x)\sqrt{x}\,dx$$

$$= 8\pi\sqrt{a}\left(\frac{ax^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right)\Bigg|_0^a = \frac{32\pi a^3}{15}.$$

$\therefore$ the volume of the solid is $\frac{32\pi a^3}{15}$ cubic units.

8 Solution



i) A slice taken perpendicular to the axis of rotation is an annulus of thickness δy with radii $r_1(y)$, $r_2(y)$, where $r_2(y) > r_1(y)$ and $r_1(y)$, $r_2(y)$ are the roots of

$$(r-b)^2+y^2=a^2$$

considered as a quadratic equation. The slice has volume

$$\delta V = \pi(r_2+r_1)(r_2-r_1)\delta y.$$

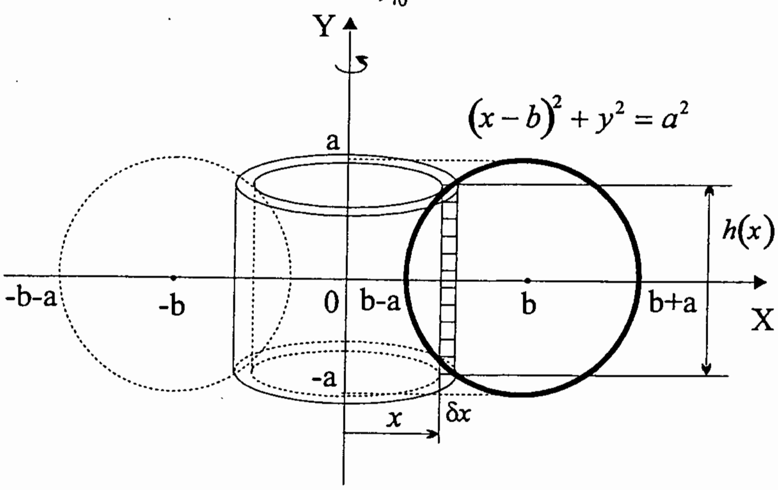
We have

$$(r-b)^2+y^2=a^2$$
$$r^2-2br+b^2-a^2+y^2=0$$
$$r_{1,2}=b\mp\sqrt{a^2-y^2}$$
$$r_2+r_1=2b$$
$$r_2-r_1=2\sqrt{a^2-y^2}$$
$$\therefore\quad \delta V = 4\pi b\sqrt{a^2-y^2}\,\delta y.$$
$$\therefore\quad V=\lim_{\delta y\to 0}\sum_{y=-a}^{a}4\pi b\sqrt{a^2-y^2}\,\delta y = 4\pi b\int_{-a}^{a}\sqrt{a^2-y^2}\,dy = 8\pi b\int_0^a\sqrt{a^2-y^2}\,dy.$$

Substitution $y=a\sin\varphi$, $dy=a\cos\varphi\, d\varphi$ gives

$$V=8\pi a^2 b\int_0^{\pi/2}\sqrt{1-\sin^2\varphi}\,\cos\varphi\,d\varphi = 8\pi a^2 b\int_0^{\pi/2}\cos^2\varphi\,d\varphi = 8\pi a^2 b\int_0^{\pi/2}\frac{1+\cos 2\varphi}{2}d\varphi$$

$$=4\pi a^2 b\left(\varphi+\frac{\sin 2\varphi}{2}\right)\Bigg|_0^{\pi/2} = 2\pi^2 a^2 b.$$



ii) The typical cylindrical shell has radii x, $x+\delta x$. Height of the shell is obtained from

$$(x-b)^2+y^2=a^2$$
$$y^2=a^2-(x-b)^2 \quad\Rightarrow$$

$$h(x)=2\sqrt{a^2-(x-b)^2}.$$

The shell has volume

$$\delta V = \pi\left[(x+\delta x)^2 - x^2\right]h(x)$$
$$= 4\pi x\sqrt{a^2-(x-b)^2}\,\delta x$$
(ignoring $(\delta x)^2$).

$$\therefore \quad V = \lim_{\delta x \to 0}\sum_{x=b-a}^{b+a} 4\pi x\sqrt{a^2-(x-b)^2}\,\delta x = 4\pi\int_{b-a}^{b+a} x\sqrt{a^2-(x-b)^2}\,dx .$$

Substitution $x = x' + b$, $dx = dx'$ gives

$$V = 4\pi\int_{-a}^{a}(x'+b)\sqrt{a^2-x'^2}\,dx' = 4\pi\int_{-a}^{a}x'\sqrt{a^2-x'^2}\,dx' + 4\pi b\int_{-a}^{a}\sqrt{a^2-x'^2}\,dx' .$$

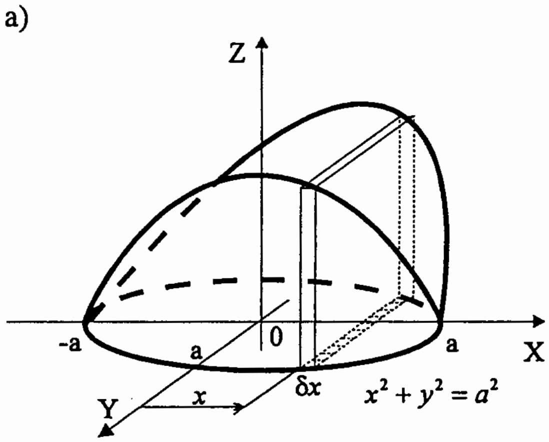
The first integral is equal to zero since the integrand is odd. Substitution $x' = \sin\varphi$, $dx' = \cos\varphi\, d\varphi$ into the second integral gives

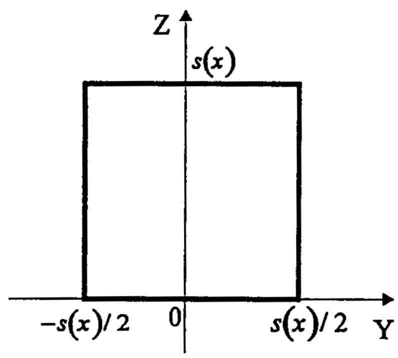
$$V = 4\pi a^2 b\int_{-\pi/2}^{\pi/2}\sqrt{1-\sin^2\varphi}\,\cos\varphi\, d\varphi = 8\pi a^2 b\int_{0}^{\pi/2}\cos^2\varphi\, d\varphi = 8\pi a^2 b\int_{0}^{\pi/2}\frac{1+\cos 2\varphi}{2}d\varphi$$
$$= 4\pi a^2 b\left(\varphi + \frac{\sin 2\varphi}{2}\right)\Bigg|_{-\pi/2}^{\pi/2} = 2\pi^2 a^2 b .$$

$\therefore$ the volume of the solid is $2\pi^2 a^2 b$ cubic units.

9 Solution

a)





The slice is a square with area of cross-section A, thickness δx.

$$A(x) = s^2(x)$$
$$s(x) = 2\sqrt{a^2-x^2}$$
$$\therefore \quad A(x) = 4\left(a^2-x^2\right).$$

The slice has volume

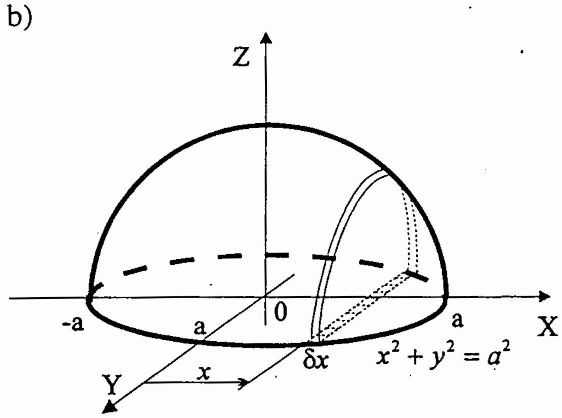
$$\delta V = A(x)\delta x = 4\left(a^2-x^2\right)\delta x .$$

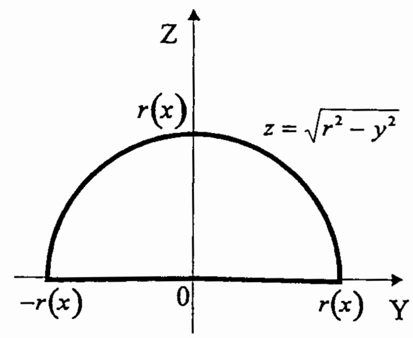
Then the volume of the solid is

$$V = \lim_{\delta x\to 0}\sum_{x=-a}^{a}4\left(a^2-x^2\right)\delta x = 4\int_{-a}^{a}\left(a^2-x^2\right)dx = 4\left(a^2x-\frac{x^3}{3}\right)\Bigg|_{-a}^{a} = \frac{16a^3}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{16a^3}{3}$ cubic units.

b)





The slice is a semicircle with area of cross-section A, thickness δx.

$$A(x)=\frac{\pi r^2(x)}{2}$$

$$r(x)=\sqrt{a^2-x^2}$$

$$\therefore \quad A(x)=\frac{\pi\left(a^2-x^2\right)}{2}.$$

The slice has volume

$$\delta V = A(x)\delta x = \frac{\pi\left(a^2-x^2\right)}{2}\delta x.$$

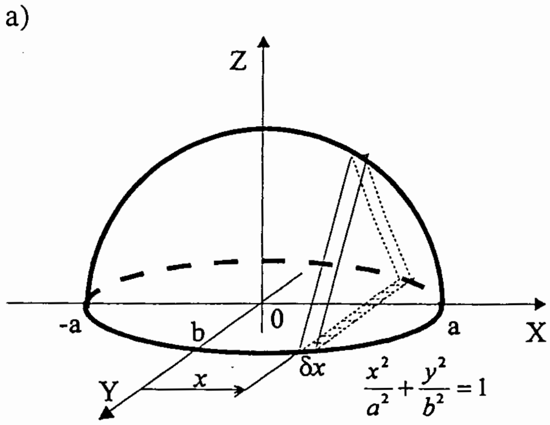
Then the volume of the solid is

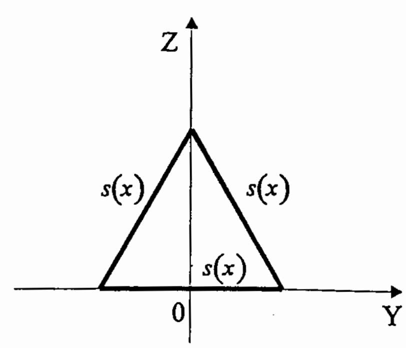
$$V=\lim_{\delta x\to 0}\sum_{x=-a}^{a}\frac{\pi\left(a^2-x^2\right)}{2}\delta x=\frac{\pi}{2}\int_{-a}^{a}\left(a^2-x^2\right)dx=\frac{\pi}{2}\left(a^2x-\frac{x^3}{3}\right)\Bigg|_{-a}^{a}=\frac{2\pi a^3}{3}.$$

$\therefore$ the volume of the solid is $\dfrac{2\pi a^3}{3}$ cubic units.

10 Solution

a)





The slice is an equilateral triangle with area of cross-section A, thickness δx.

$$A(x)=\frac{\sqrt{3}\,s^2(x)}{4}$$

$$s(x)=2b\sqrt{1-\frac{x^2}{a^2}}$$

$$\therefore\quad A(x)=\sqrt{3}b^2\left(1-\frac{x^2}{a^2}\right).$$

The slice has volume

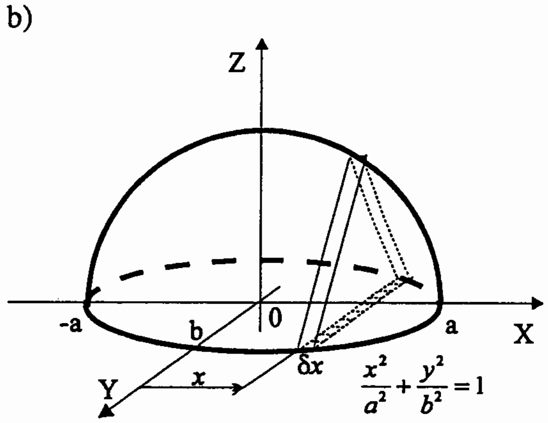
$$\delta V=A(x)\delta x=\sqrt{3}b^2\left(1-\frac{x^2}{a^2}\right)\delta x.$$

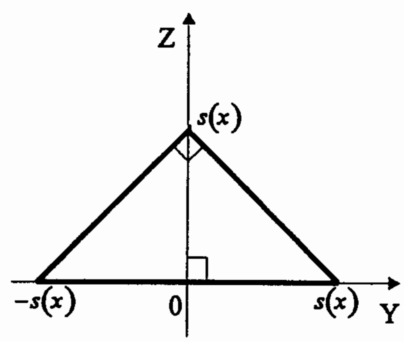
Then the volume of the solid is

$$V=\lim_{\delta x\to 0}\sum_{x=-a}^{a}\sqrt{3}b^2\left(1-\frac{x^2}{a^2}\right)\delta x=\sqrt{3}b^2\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx=\sqrt{3}b^2\left(x-\frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a}=\frac{4ab^2}{\sqrt{3}}.$$

$\therefore$ the volume of the solid is $\frac{4ab^2}{\sqrt{3}}$ cubic units.

b)





The slice is an isosceles right-angled triangle with area of cross-section A, thickness δx.

$$A(x)=s^2(x)$$

$$s(x)=b\sqrt{1-\frac{x^2}{a^2}}$$

$$\therefore\quad A(x)=b^2\left(1-\frac{x^2}{a^2}\right).$$

The slice has volume

$$\delta V=A(x)\delta x=b^2\left(1-\frac{x^2}{a^2}\right)\delta x.$$

Then the volume of the solid is

$$V=\lim_{\delta x\to 0}\sum_{x=-a}^{a}b^2\left(1-\frac{x^2}{a^2}\right)\delta x=b^2\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx=b^2\left(x-\frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a}=\frac{4ab^2}{3}.$$

$\therefore$ the volume of the solid is $\frac{4ab^2}{3}$ cubic units.