7SD Solutions Series

Worked Solutions to Popular Mathematics Texts

Suggested Worked Solutions to

"4 Unit Mathematics"

(Text book for the NSW HSC by D. Arnold and G. Arnold)

Chapter 3
Conics

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Solutions are to "4 Unit Mathematics"

[by D. Arnold and G. Arnold (1993), ISBN 0 340 54335 3]

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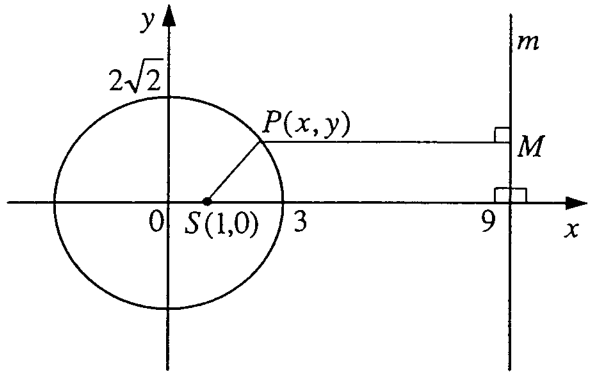
Exercise 3.1

1 Solution

(a) The locus of a variable point $P(x,y)$ is the ellipse with focus at $S(1,0)$, directrix $m: x=9$ and eccentricity $e=\frac{1}{3}$. Let M be the foot of the perpendicular from P to m. Then M has coordinates $(9,y)$.

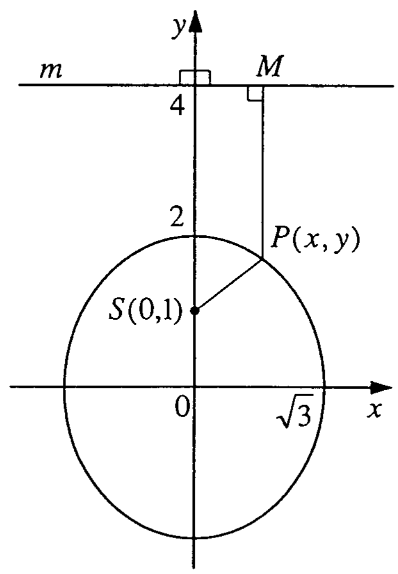


$$PS = e\cdot PM \Rightarrow (x-1)^2+y^2=\left(\frac{1}{3}\right)^2(x-9)^2$$

$$x^2\left(1-\frac{1}{9}\right)+y^2=9-1.$$

Therefore the Cartesian equation of the ellipse is $\dfrac{x^2}{9}+\dfrac{y^2}{8}=1$.

(b) The locus of a variable point $P(x,y)$ is the ellipse with focus at $S(0,1)$, directrix $m: y=4$ and eccentricity $e=\frac{1}{2}$. Let M be the foot of the perpendicular from P to m. Then M has coordinates $(x,4)$.

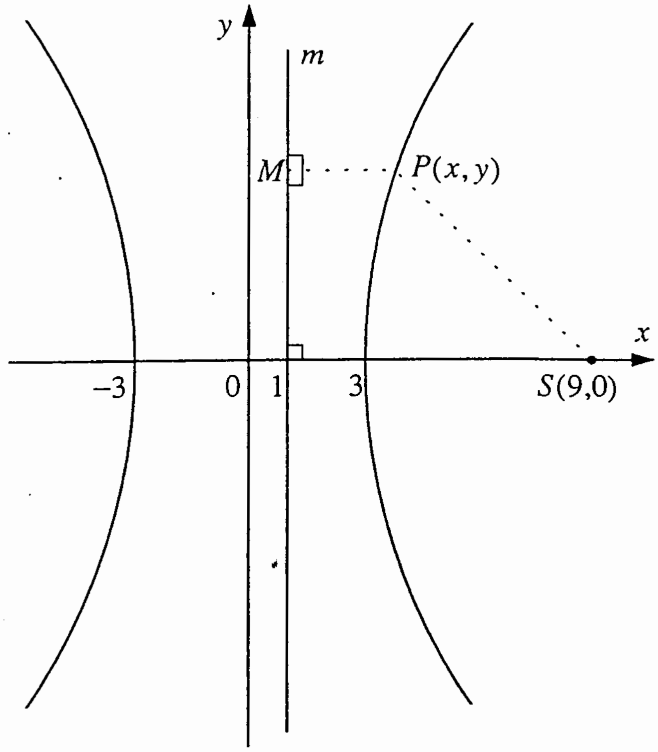


$$PS = e\cdot PM \Rightarrow x^2+(y-1)^2=\left(\frac{1}{2}\right)^2(y-4)^2$$

$$x^2+y^2\left(1-\frac{1}{4}\right)=4-1.$$

Therefore the Cartesian equation of the ellipse is $\dfrac{x^2}{3}+\dfrac{y^2}{4}=1$.

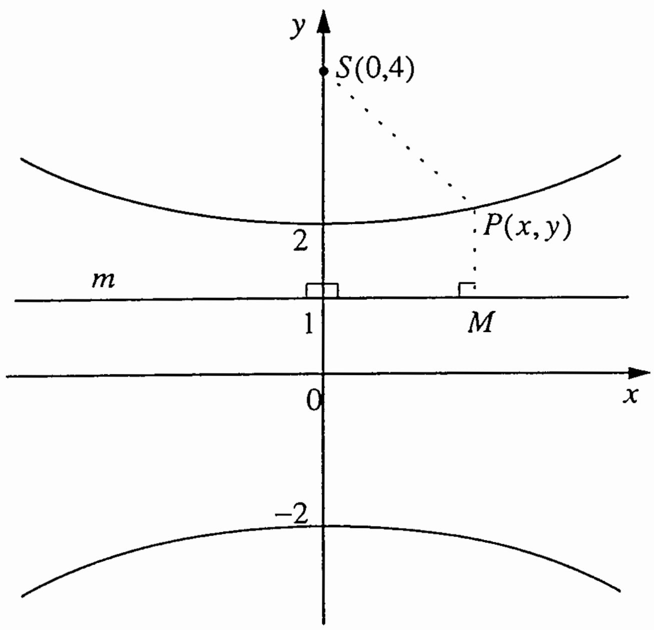
(c) The locus of a variable point $P(x, y)$ is the hyperbola with focus at $S(9,0)$, directrix $m: x = 1$ and eccentricity $e = 3$. Let M be the foot of the perpendicular from P to m. Then M has coordinates $(1, y)$.



$$PS = e \cdot PM \Rightarrow$$
$$(x-9)^2 + y^2 = 3^2(x-1)^2$$
$$x^2(1-9) + y^2 = 9 - 81.$$

Therefore the Cartesian equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{72} = 1$.

(d) The locus of a variable point $P(x, y)$ is the hyperbola with focus at $S(0,4)$, directrix $m: y = 1$ and eccentricity $e = 2$. Let M be the foot of the perpendicular from P to m. Then M has coordinates $(x,1)$.

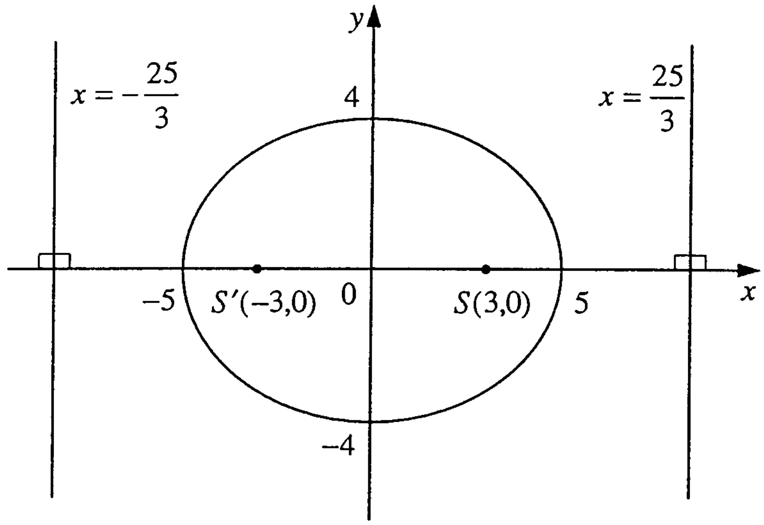


$$PS = e \cdot PM \Rightarrow$$
$$x^2 + (y-4)^2 = 2^2(y-1)^2$$
$$x^2 + y^2(1-4) = 4 - 16.$$

Therefore the Cartesian equation of the hyperbola is $\frac{y^2}{4} - \frac{x^2}{12} = 1$.

2 Solution

(a) $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$



$a=5,\ b=4 \Rightarrow b<a$

$b^2=a^2(1-e^2)$

eccentricity:

$e=\sqrt{1-\dfrac{16}{25}}=\dfrac{3}{5}$

foci:

$(\pm ae,0) \Rightarrow (\pm 3,0)$

directrices: $x=\pm\dfrac{a}{e} \Rightarrow x=\pm\dfrac{25}{3}$

(b) $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$

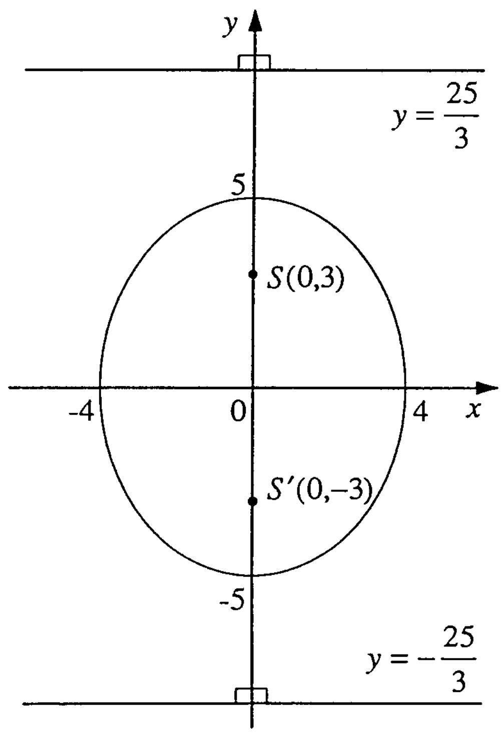
$a=4,\ b=5 \Rightarrow b>a$

$a^2=b^2(1-e^2)$

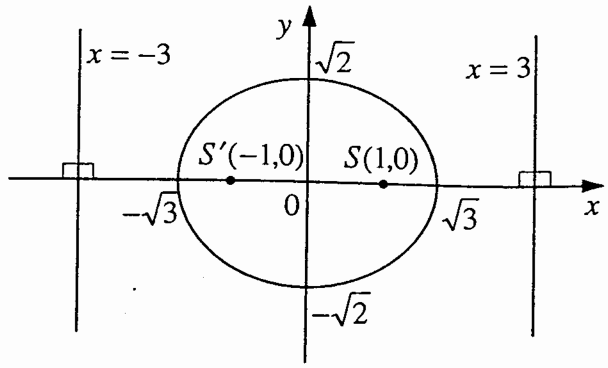
eccentricity: $e=\sqrt{1-\dfrac{16}{25}}=\dfrac{3}{5}$

foci: $(0,\pm be) \Rightarrow (0,\pm 3)$

directrices: $y=\pm\dfrac{b}{e} \Rightarrow y=\pm\dfrac{25}{3}$



(c) $\dfrac{x^2}{3}+\dfrac{y^2}{2}=1$



$a=\sqrt{3},\ b=\sqrt{2}\Rightarrow b<a$

$b^2=a^2(1-e^2)$

eccentricity: $e=\sqrt{1-\dfrac{2}{3}}=\dfrac{1}{\sqrt{3}}$

foci: $(\pm ae,0)\Rightarrow(\pm1,0)$

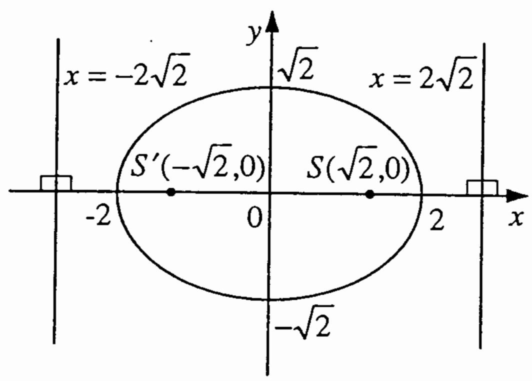
directrices: $x=\pm\dfrac{a}{e}\Rightarrow x=\pm3$

(d) $x^2+2y^2=4$

$\dfrac{x^2}{4}+\dfrac{y^2}{2}=1$

$a=2,\ b=\sqrt{2}\Rightarrow b<a$

$b^2=a^2(1-e^2)$



eccentricity: $e=\sqrt{1-\dfrac{2}{4}}=\dfrac{1}{\sqrt{2}}$

foci: $(\pm ae,0)\Rightarrow(\pm\sqrt{2},0)$

directrices; $x=\pm\dfrac{a}{e}\Rightarrow x=\pm2\sqrt{2}$

3 Solution

(a) $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$

$a=3,\ b=4$

$b^2=a^2(e^2-1)$

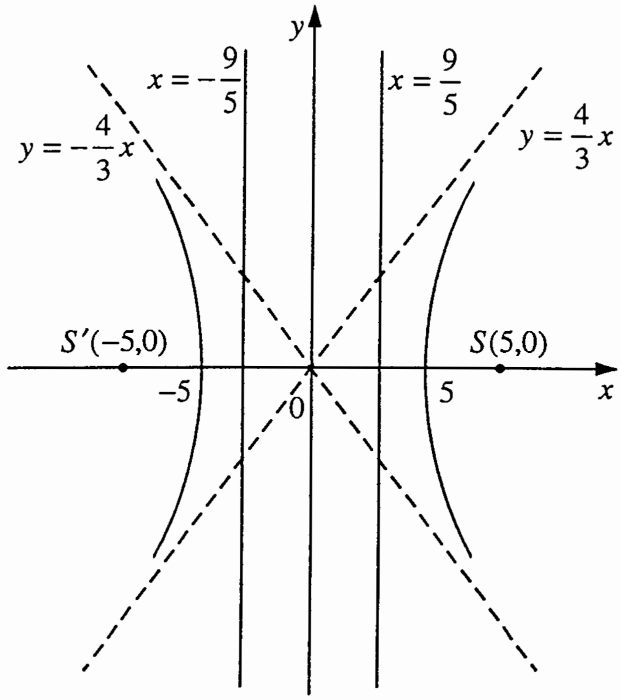
eccentricity: $e=\sqrt{1+\dfrac{16}{9}}=\dfrac{5}{3}$

foci: $(\pm ae,0)\Rightarrow(\pm 5,0)$

directrices: $x=\pm\dfrac{a}{e}\Rightarrow x=\pm\dfrac{9}{5}$

asymptotes:

$y=\pm\dfrac{b}{a}x\Rightarrow y=\pm\dfrac{4}{3}x$



(b) $\dfrac{y^2}{16}-\dfrac{x^2}{9}=1$

$a=3,\ b=4$

$a^2=b^2(e^2-1)$

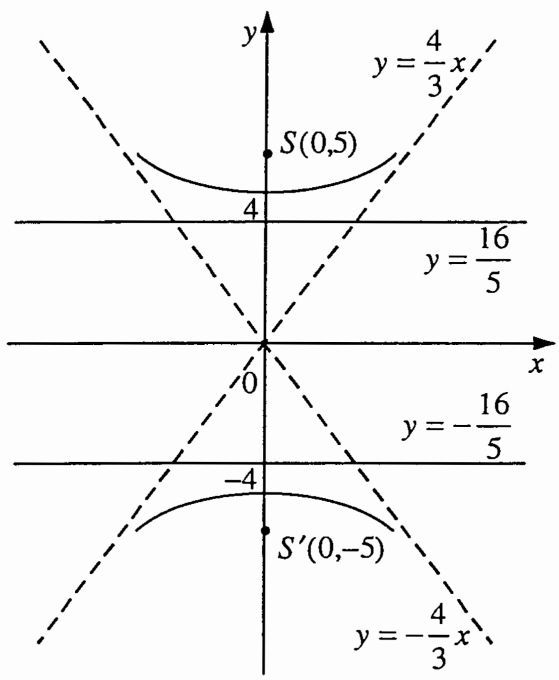
eccentricity: $e=\sqrt{1+\dfrac{9}{16}}=\dfrac{5}{4}$

foci: $(0,\pm be)\Rightarrow(0,\pm 5)$

directrices: $y=\pm\dfrac{b}{e}\Rightarrow y=\pm\dfrac{16}{5}$

asymptotes: $x=\pm\dfrac{a}{b}y\Rightarrow x=\pm\dfrac{3}{4}y$

$\Rightarrow y=\pm\dfrac{4}{3}x$



(c) $$\frac{x^2}{2}-\frac{y^2}{4}=1$$

$$a=\sqrt{2},\ b=2$$

$$b^2=a^2(e^2-1)$$

eccentricity: $e=\sqrt{1+\frac{4}{2}}=\sqrt{3}$

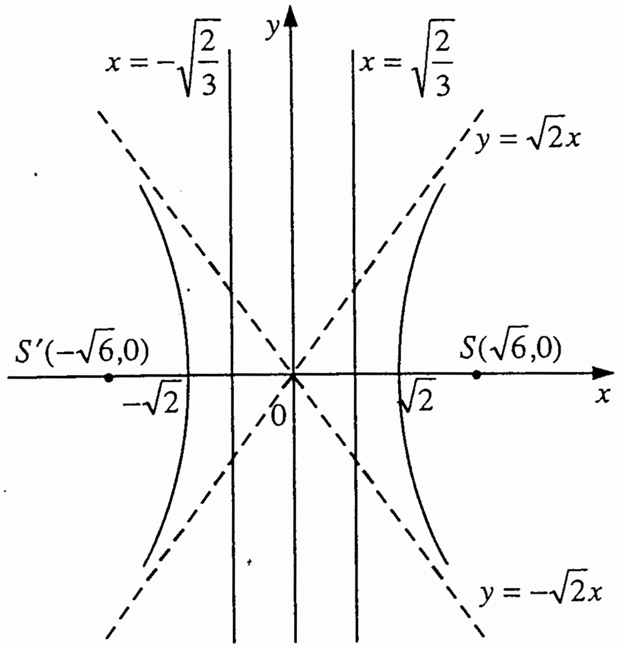
foci: $(\pm ae,0)\Rightarrow(\pm\sqrt{6},0)$

directrices:

$$x=\pm\frac{a}{e}\Rightarrow x=\pm\sqrt{\frac{2}{3}}$$

asymptotes:

$$y=\pm\frac{b}{a}x\Rightarrow y=\pm\sqrt{2}x$$



(d) $$x^2-y^2=4$$

$$\frac{x^2}{4}-\frac{y^2}{4}=1$$

$$a=2,\ b=2$$

$$b^2=a^2(e^2-1)$$

eccentricity:

$$e=\sqrt{1+\frac{4}{4}}=\sqrt{2}$$

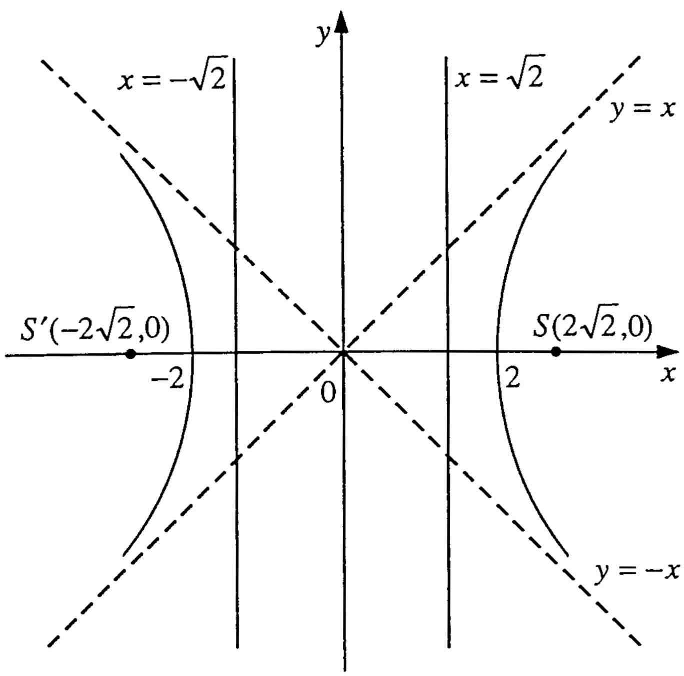
foci:

$$(\pm ae,0)\Rightarrow(\pm2\sqrt{2},0)$$

directrices:

$$x=\pm\frac{a}{e}\Rightarrow x=\pm\sqrt{2}$$

asymptotes: $y=\pm\frac{b}{a}x\Rightarrow y=\pm x$



4 Solution

(a) We have the eccentricity $e=\frac{4}{5}$ and the foci $(\pm 4,0)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. But the coordinates of the foci are $(\pm ae,0)$. Therefore $a=4\cdot\frac{5}{4}=5$. Then $b^2=a^2(1-e^2)=25\cdot\left(1-\frac{16}{25}\right)=9$. Hence the Cartesian equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

(b) We have the eccentricity $e=\frac{2}{3}$ and the directrices $x=\pm 9$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. But the directrices have equations $x=\pm\frac{a}{e}$. Therefore $a=9\cdot\frac{2}{3}=6$. Then $b^2=a^2(1-e^2)=36\cdot\left(1-\frac{4}{9}\right)=20$. Hence the Cartesian equation of the ellipse is $\frac{x^2}{36}+\frac{y^2}{20}=1$.

5 Solution

(a) We have the eccentricity $e=\frac{5}{4}$ and the foci $(\pm 5,0)$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. But the coordinates of the foci are $(\pm ae,0)$. Therefore $a=5\cdot\frac{4}{5}=4$. Then $b^2=a^2(e^2-1)=16\cdot\left(\frac{25}{16}-1\right)=9$. Hence the Cartesian equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$.

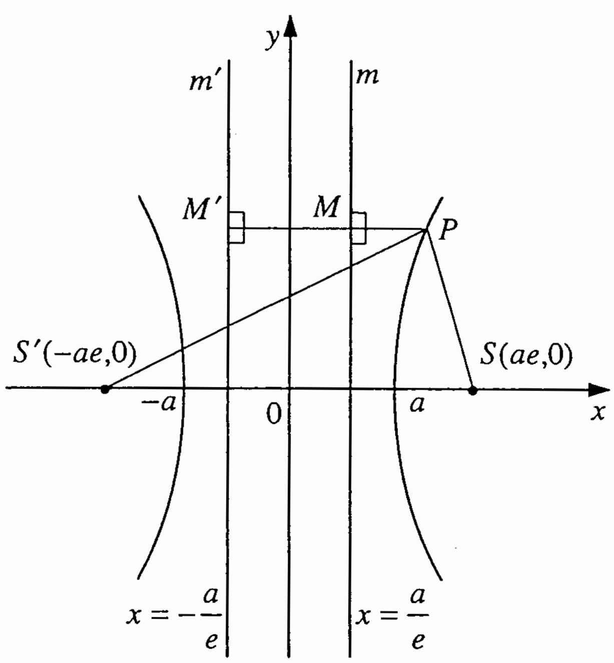
(b) We have the eccentricity $e=\frac{3}{2}$ and the directrices $x=\pm 4$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. But the directrices have equations $x=\pm\frac{a}{e}$. Therefore $a=4\cdot\frac{3}{2}=6$. Then $b^2=a^2(e^2-1)=36\cdot\left(\frac{9}{4}-1\right)=45$. Hence the Cartesian equation of the hyperbola is $\dfrac{x^2}{36} - \dfrac{y^2}{45} = 1$.

6 Solution

The equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Thus we need to find the parameters a and b. Since the foci are on the x-axes, their coordinates are $(\pm ae, 0)$. Therefore the distance between the foci is $2ae = 4$. The equations of the directrices are $x = \pm\dfrac{a}{e}$. Hence the distance between the directrices is $2 \cdot \dfrac{a}{e} = 16$. Thus we have two equations $ae = 2$ and $\dfrac{a}{e} = 8$. From the first equation we get $e = \dfrac{2}{a}$. Substituting the expression for the e to the second equation we obtain $a^2 = 16$. Therefore $a = 4$ and $e = \dfrac{2}{4} = \dfrac{1}{2}$. Then $b^2 = a^2(1 - e^2) = 16 \cdot \left(1 - \dfrac{1}{4}\right) = 12$. Hence the Cartesian equation of the ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$.

7 Solution

Let m and m' be the directrices of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Then for P on the curve, both $PS = e \cdot PM$ and $PS' = e \cdot PM'$, where M and M' are the feet of the perpendiculars from P to m and m' respectively. Therefore $|PS - PS'| = e|PM - PM'| = eMM'$. Thus $|PS - PS'| = 2a$.



For the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{72} = 1$

$a = 3$. Hence $|PS - PS'| = 6$. Since $b^2 = 72$, $e = \sqrt{\frac{b^2}{a^2} + 1} = \sqrt{\frac{72}{9} + 1} = 3$. Therefore the coordinates of the foci are $(\pm 9, 0)$.

(a) If $PS = 2$, then $|PS' - 2| = 6$. Thus $PS' = 8$. We see that $PS + PS' = 10$. But $MM' = 18$. Hence there is no such point P on the hyperbola.

(b) If $PS = 8$, then $|PS' - 8| = 6$. Thus $PS' = 14$.

Exercise 3.2

1 Solution

(a) Cartesian equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$. Hence $a=4$ and $b=3$. Therefore the ellipse has parametric equations $x=4\cos\theta$ and $y=3\sin\theta$, $-\pi<\theta\le\pi$.

(b) Cartesian equation of the ellipse is $x^2+4y^2=4$. Then $\frac{x^2}{4}+\frac{y^2}{1}=1$. Hence $a=2$ and $b=1$. Therefore the ellipse has parametric equations $x=2\cos\theta$ and $y=\sin\theta$, $-\pi<\theta\le\pi$.

(c) Cartesian equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{25}=1$. Hence $a=4$ and $b=5$. Therefore the hyperbola has parametric equations $x=4\sec\theta$ and $y=5\tan\theta$, $-\pi<\theta\le\pi$, $\theta\ne\pm\frac{\pi}{2}$.

(d) Cartesian equation of the hyperbola is $x^2-y^2=4$. Then $\frac{x^2}{4}-\frac{y^2}{4}=1$. Hence $a=2$ and $b=2$. Therefore the hyperbola has parametric equations $x=2\sec\theta$ and $y=2\tan\theta$, $-\pi<\theta\le\pi$, $\theta\ne\pm\frac{\pi}{2}$.

2 Solution

(a) The ellipse has parametric equations $x=3\cos\theta$, $y=2\sin\theta$. Therefore $\frac{x^2}{9}+\frac{y^2}{4}=\cos^2\theta+\sin^2\theta=1$. Hence the Cartesian equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$.

(b) The ellipse has parametric equations $x=5\cos\theta$, $y=4\sin\theta$. Therefore $\frac{x^2}{25}+\frac{y^2}{16}=\cos^2\theta+\sin^2\theta=1$. Hence the Cartesian equation of the ellipse is

$\frac{x^2}{25}+\frac{y^2}{16}=1$.

(c) The hyperbola has parametric equations $x=3\sec\theta$, $y=4\tan\theta$. Therefore $\frac{x^2}{9}-\frac{y^2}{16}=\sec^2\theta-\tan^2\theta=1$. Hence the Cartesian equation of the hyperbola is $\frac{x^2}{9}-\frac{y^2}{16}=1$.

(d) The hyperbola has parametric equations $x=2\sec\theta$, $y=5\tan\theta$. Therefore $\frac{x^2}{4}-\frac{y^2}{25}=\sec^2\theta-\tan^2\theta=1$. Hence the Cartesian equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{25}=1$.

3 Solution

The equation of the chord PQ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$, where P, Q have parameters θ, ϕ. We have $\phi=\pi+\theta$. Hence the equation of the chord PQ transforms into $\frac{x}{a}\cos\left(\frac{2\theta+\pi}{2}\right)+\frac{y}{b}\sin\left(\frac{2\theta+\pi}{2}\right)=\cos\left(\frac{-\pi}{2}\right)$. Thus $-\frac{x}{a}\sin\theta+\frac{y}{b}\cos\theta=0$. Therefore (0,0) lies on the chord PQ.

4 Solution

The equation of the chord PQ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$, where P, Q have parameters θ, ϕ. We have $\phi=\pi-\theta$. Hence the equation of the chord PQ transforms into $\frac{x}{a}\cos\left(\frac{2\theta-\pi}{2}\right)-\frac{y}{b}\sin\left(\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}\right)$. Thus $\frac{x}{a}\sin\theta-\frac{y}{b}=0$. Therefore (0,0) lies on the chord PQ.

5 Solution

(a) Chord PQ has equation $x = ae$, P has coordinates $(a\cos\theta, b\sin\theta)$. Hence $a\cos\theta = ae$. Thus $\cos\theta = e$.

(b) Length of the chord PQ is $|b\sin\theta - b\sin(-\theta)| = 2b|\sin\theta| = 2b\sqrt{1-\cos^2\theta} = 2b\sqrt{1-e^2}$. But for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we have $b^2 = a^2(1-e^2)$. Therefore the length of the chord PQ is $2b \cdot \frac{b}{a} = \frac{2b^2}{a}$.

6 Solution

(a) Length of PS is $\sqrt{(a\sec\theta - ae)^2 + (b\tan\theta)^2} = \sqrt{a^2(\sec\theta - e)^2 + b^2\tan^2\theta}$. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we have $b^2 = a^2(e^2-1)$. Therefore the length of PS is

$$\sqrt{a^2(\sec\theta - e)^2 + a^2(e^2-1)\tan^2\theta} = a\sqrt{\sec^2\theta - 2e\sec\theta + e^2 + e^2\tan^2\theta - \tan^2\theta} =$$
$$a\sqrt{e^2(1+\tan^2\theta) - 2e\sec\theta + (\sec^2\theta - \tan^2\theta)} = a\sqrt{e^2\sec^2\theta - 2e\sec\theta + 1} = a\sqrt{(e\sec\theta - 1)^2}$$

Hence the length of PS is $a|e\sec\theta - 1|$.

Length of PS' is $\sqrt{(a\sec\theta + ae)^2 + (b\tan\theta)^2} = \sqrt{a^2(\sec\theta + e)^2 + b^2\tan^2\theta}$. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we have $b^2 = a^2(e^2-1)$. Therefore the length of PS' is

$$\sqrt{a^2(\sec\theta + e)^2 + a^2(e^2-1)\tan^2\theta} = a\sqrt{\sec^2\theta + 2e\sec\theta + e^2 + e^2\tan^2\theta - \tan^2\theta} =$$
$$a\sqrt{e^2(1+\tan^2\theta) + 2e\sec\theta + (\sec^2\theta - \tan^2\theta)} = a\sqrt{e^2\sec^2\theta + 2e\sec\theta + 1} = a\sqrt{(e\sec\theta + 1)^2}$$

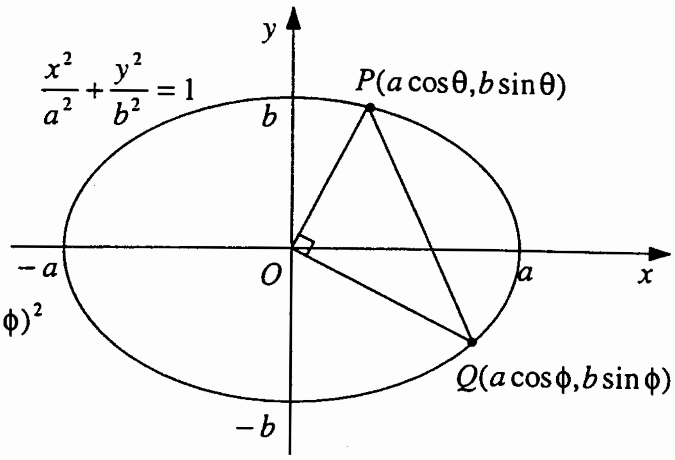
Hence the length of PS' is $a|e\sec\theta + 1|$.

(b) If P lies on the right-hand branch of the hyperbola, then $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Since for hyperbola $e > 1$, $PS = a(e\sec\theta - 1)$ and $PS' = a(e\sec\theta + 1)$. Therefore $PS - PS' = -2a$. If P lies on the left-hand branch of the hyperbola, then $-\pi < \theta < -\frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$. Since for hyperbola $e > 1$, $PS = -a(e\sec\theta - 1)$ and

$PS' = -a(e\sec\theta + 1)$. Therefore $PS - PS' = +2a$. Hence $|PS - PS'| = 2a$.

7 Solution

(a) POQ is a right-angled triangle. Therefore $OP^2 + OQ^2 = PQ^2$.



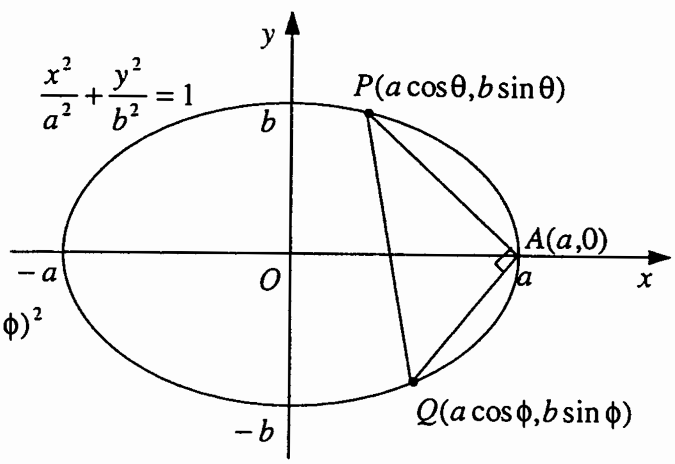
$$a^2\cos^2\theta + b^2\sin^2\theta + a^2\cos^2\phi + b^2\sin^2\phi = a^2(\cos\theta - \cos\phi)^2 + b^2(\sin\theta - \sin\phi)^2$$

Then

$$0 = -2a^2\cos\theta\cos\phi - 2b^2\sin\theta\sin\phi$$

$$\therefore \tan\theta\tan\phi = -\frac{a^2}{b^2}$$

(b) PAQ is a right-angled triangle. Therefore $AP^2 + AQ^2 = PQ^2$.



$$a^2(\cos\theta - 1)^2 + b^2\sin^2\theta + a^2(\cos\phi - 1)^2 + b^2\sin^2\phi = a^2(\cos\theta - \cos\phi)^2 + b^2(\sin\theta - \sin\phi)^2$$

Then

$$-2a^2\cos\theta + a^2 - 2a^2\cos\phi + a^2 = -2a^2\cos\theta\cos\phi - 2b^2\sin\theta\sin\phi,$$

$$\cos\theta + \cos\phi - 1 - \cos\theta\cos\phi = \frac{b^2}{a^2}\sin\theta\sin\phi,$$

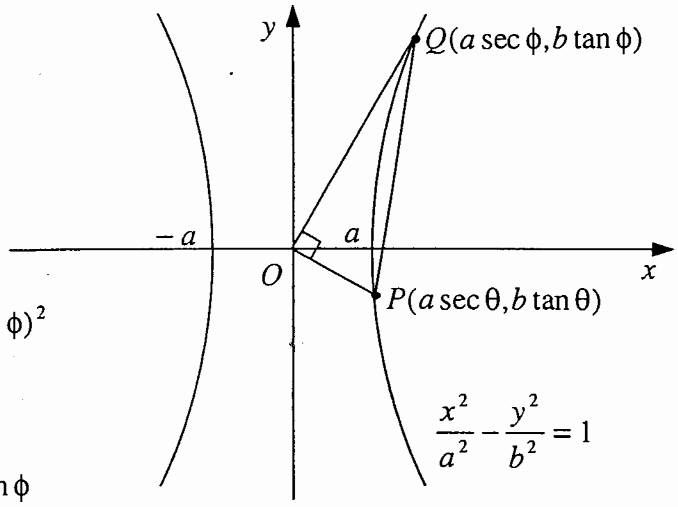
$$\left(1 - 2\sin^2\frac{\theta}{2}\right) + \left(1 - 2\sin^2\frac{\phi}{2}\right) - 1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\phi}{2}\right) = \frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right),$$

$$-4\sin^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right).$$

Hence $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$.

8 Solution

(a) POQ is a right-angled triangle. Therefore $OP^2+OQ^2=PQ^2$.



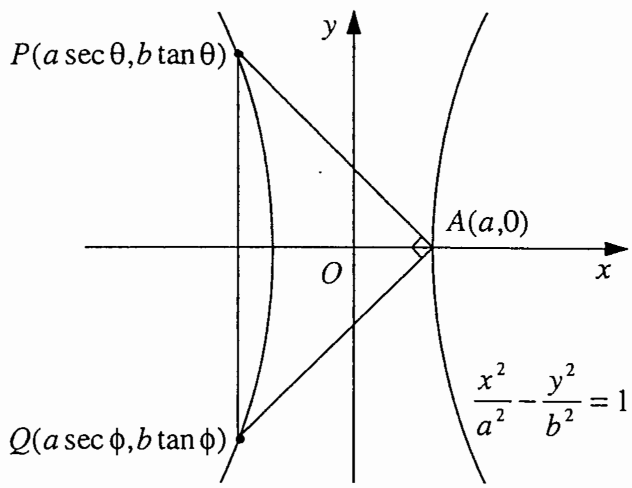
$a^2\sec^2\theta+b^2\tan^2\theta+$
$a^2\sec^2\phi+b^2\tan^2\phi=$
$a^2(\sec\theta-\sec\phi)^2+b^2(\tan\theta-\tan\phi)^2$

Then

$0=-2a^2\sec\theta\sec\phi-2b^2\tan\theta\tan\phi$

$\therefore \sin\theta\sin\phi=-\frac{a^2}{b^2}$

(b) PAQ is a right-angled triangle. Therefore $AP^2+AQ^2=PQ^2$.



$a^2(\sec\theta-1)^2+b^2\tan^2\theta+$
$a^2(\sec\phi-1)^2+b^2\tan^2\phi=$
$a^2(\sec\theta-\sec\phi)^2+b^2(\tan\theta-\tan\phi)^2$

Then

$-2a^2\sec\theta+a^2-2a^2\sec\phi+a^2=$
$-2a^2\sec\theta\sec\phi-2b^2\tan\theta\tan\phi$

,

$$\cos\theta+\cos\phi-1-\cos\theta\cos\phi=\frac{b^2}{a^2}\sin\theta\sin\phi,$$

$$\left(1-2\sin^2\frac{\theta}{2}\right)+\left(1-2\sin^2\frac{\phi}{2}\right)-1-\left(1-2\sin^2\frac{\theta}{2}\right)\left(1-2\sin^2\frac{\phi}{2}\right)=$$

$$\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right)$$

$$-4\sin^2\frac{\theta}{2}\sin^2\frac{\theta}{2}=\frac{b^2}{a^2}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\left(2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\right).$$

Hence $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$.

9 Solution

(a) If PQ is a focal chord through $S(ae,0)$, then $e\cos\left(\frac{\theta-\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$.

Expanding both cosines gives $(e-1)\cos\frac{\theta}{2}\cos\frac{\phi}{2}=-(e+1)\sin\frac{\theta}{2}\sin\frac{\phi}{2}$. Hence $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{1-e}{1+e}$. Similarly, if PQ is a focal chord through $S'(-ae,0)$, Then replacing e by $-e$, $\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{1+e}{1-e}$.

(b) $\frac{x^2}{3}-\frac{y^2}{9}=1\Rightarrow a=\sqrt{3}$ and $b=3$, $\therefore P\left(2\sqrt{3},3\sqrt{3}\right)\equiv P\left(\sqrt{3}\sec\frac{\pi}{3},3\tan\frac{\pi}{3}\right)$.

Also $b^2=a^2(e^2-1)$ $\therefore e=\sqrt{\left(1+\frac{9}{3}\right)}=2$. P has parameter $\frac{\pi}{3}$. Let Q has parameter ϕ. Hence

$$\tan\frac{\pi}{6}\tan\frac{\phi}{2}=\frac{1-2}{1+2},\qquad\text{or}\quad\tan\frac{\pi}{6}\tan\frac{\phi}{2}=\frac{1+2}{1-2},$$

$$\therefore\tan\frac{\phi}{2}=-\frac{1}{\sqrt{3}},\qquad\tan\frac{\phi}{2}=-3\sqrt{3},$$

$$\sec\phi=\frac{1+\frac{1}{3}}{1-\frac{1}{3}}=2,\qquad\sec\phi=\frac{1+27}{1-27}=-\frac{14}{13},$$

$$\text{and }\tan\phi=\frac{2\left(-\frac{1}{\sqrt{3}}\right)}{1-\frac{1}{3}}=-\sqrt{3}.\qquad\text{and }\tan\phi=\frac{2\left(-3\sqrt{3}\right)}{1-27}=\frac{3\sqrt{3}}{13}.$$

Q has coordinates $(\sqrt{3}\sec\phi,3\tan\phi)\Rightarrow Q\left(2\sqrt{3},-3\sqrt{3}\right)$ or $Q\left(-\frac{14\sqrt{3}}{13},\frac{9\sqrt{3}}{13}\right)$.

Exercise 3.3

1 Solution

(a) The tangent to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ at the point $(3,2)$ has equation $\frac{3x}{15}+\frac{2y}{10}=1 \Rightarrow x+y=5$. The normal to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ at the point $(3,2)$ has equation $\frac{15x}{3}-\frac{10y}{2}=15-10 \Rightarrow x-y=1$.

(b) $3x^2+4y^2=48 \Rightarrow \frac{x^2}{16}+\frac{y^2}{12}=1$. The tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ at the point $(2,-3)$ has equation $\frac{2x}{16}+\frac{-3y}{12}=1 \Rightarrow x-2y=8$. The normal to the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ at the point $(2,-3)$ has equation $\frac{16x}{2}-\frac{12y}{-3}=16-12 \Rightarrow 2x+y=1$.

(c) The tangent to the hyperbola $\frac{x^2}{6}-\frac{y^2}{8}=1$ at the point $(3,2)$ has equation $\frac{3x}{6}-\frac{2y}{8}=1 \Rightarrow 2x-y=4$. The normal to the hyperbola $\frac{x^2}{6}-\frac{y^2}{8}=1$ at the point $(3,2)$ has equation $\frac{6x}{3}+\frac{8y}{2}=6+8 \Rightarrow x+2y=7$.

(d) $9x^2-2y^2=18 \Rightarrow \frac{x^2}{2}-\frac{y^2}{9}=1$. The tangent to the hyperbola $\frac{x^2}{2}-\frac{y^2}{9}=1$ at the point $(2,-3)$ has equation $\frac{2x}{2}-\frac{-3y}{9}=1 \Rightarrow 3x+y=3$. The normal to the hyperbola $\frac{x^2}{2}-\frac{y^2}{9}=1$ at the point $(2,-3)$ has equation $\frac{2x}{2}+\frac{9y}{-3}=2+9 \Rightarrow x-3y=11$.

2 Solution

(a) The tangent to the ellipse $x=6\cos\theta,\ y=2\sin\theta$ at the point where $\theta=\frac{\pi}{6}$ has equation $\frac{x\cos\frac{\pi}{6}}{6}+\frac{y\sin\frac{\pi}{6}}{2}=1 \Rightarrow \sqrt{3}x+3y=12$. The normal to the ellipse

$x = 6\cos\theta,\ y = 2\sin\theta$ at the point where $\theta = \frac{\pi}{6}$ has equation

$$\frac{6x}{\cos\frac{\pi}{6}} - \frac{2y}{\sin\frac{\pi}{6}} = 36 - 4 \Rightarrow 3x - \sqrt{3}y = 8\sqrt{3}.$$

(b) The tangent to the ellipse $x = 4\cos\theta,\ y = 2\sin\theta$ at the point where $\theta = -\frac{\pi}{4}$ has equation $\dfrac{x\cos\left(-\frac{\pi}{4}\right)}{4} + \dfrac{y\sin\left(-\frac{\pi}{4}\right)}{2} = 1 \Rightarrow x - 2y = 4\sqrt{2}$. The normal to the ellipse $x = 4\cos\theta,\ y = 2\sin\theta$ at the point where $\theta = -\frac{\pi}{4}$ has equation

$$\frac{4x}{\cos\left(-\frac{\pi}{4}\right)} - \frac{2y}{\sin\left(-\frac{\pi}{4}\right)} = 16 - 4 \Rightarrow 2x + y = 3\sqrt{2}.$$

(c) The tangent to the hyperbola $x = 2\sec\theta,\ y = 3\tan\theta$ at the point where $\theta = \frac{\pi}{3}$ has equation $\dfrac{x\sec\frac{\pi}{3}}{2} - \dfrac{y\tan\frac{\pi}{3}}{3} = 1 \Rightarrow 3x - \sqrt{3}y = 3$. The normal to the hyperbola $x = 2\sec\theta,\ y = 3\tan\theta$ at the point where $\theta = \frac{\pi}{3}$ has equation

$$\frac{2x}{\sec\frac{\pi}{3}} + \frac{3y}{\tan\frac{\pi}{3}} = 4 + 9 \Rightarrow x + \sqrt{3}y = 13.$$

(d) The tangent to the hyperbola $x = 2\sec\theta,\ y = 4\tan\theta$ at the point where $\theta = -\frac{\pi}{4}$ has equation $\dfrac{x\sec\left(-\frac{\pi}{4}\right)}{2} - \dfrac{y\tan\left(-\frac{\pi}{4}\right)}{4} = 1 \Rightarrow 2\sqrt{2}x + y = 4$. The normal to the hyperbola $x = 2\sec\theta,\ y = 4\tan\theta$ at the point where $\theta = -\frac{\pi}{4}$ has equation

$$\frac{2x}{\sec\left(-\frac{\pi}{4}\right)} + \frac{4y}{\tan\left(-\frac{\pi}{4}\right)} = 4 + 16 \Rightarrow x - 2\sqrt{2}y = 10\sqrt{2}.$$

3 Solution

(a) The chord of contact of tangents from the point (5,4) to the ellipse $\frac{x^2}{15}+\frac{y^2}{10}=1$ has equation $\frac{5x}{15}+\frac{4y}{10}=1 \Rightarrow 5x+6y=15$.

(b) $3x^2+4y^2=48 \Rightarrow \frac{x^2}{16}+\frac{y^2}{12}=1$. The chord of contact of tangents from the point (6,4) to the ellipse $\frac{x^2}{16}+\frac{y^2}{12}=1$ has equation $\frac{6x}{16}+\frac{4y}{12}=1 \Rightarrow 9x+8y=24$.

(c) The chord of contact of tangents from the point (1,2) to the hyperbola $\frac{x^2}{6}-\frac{y^2}{8}=1$ has equation $\frac{x}{6}-\frac{2y}{8}=1 \Rightarrow 2x-3y=12$.

(d) $9x^2-2y^2=18 \Rightarrow \frac{x^2}{2}-\frac{y^2}{9}=1$. The chord of contact of tangents from the point (1,2) to the hyperbola $\frac{x^2}{2}-\frac{y^2}{9}=1$ has equation $\frac{x}{2}-\frac{2y}{9}=1 \Rightarrow 9x-4y=18$.

4 Solution

The normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. Point X has coordinates $\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$ and point Y has coordinates $\left(0, \frac{b^2-a^2}{b}\sin\theta\right)$. Hence

$$PX^2=\left(a-\frac{a^2-b^2}{a}\right)^2\cos^2\theta+b^2\sin^2\theta=\frac{b^4}{a^2}\cos^2\theta+b^2\sin^2\theta=\frac{b^2}{a^2}\left(b^2\cos^2\theta+a^2\sin^2\theta\right)$$

$$PY^2=a^2\cos^2\theta+\left(b-\frac{b^2-a^2}{b}\right)^2\sin^2\theta=a^2\cos^2\theta+\frac{a^4}{b^2}\sin^2\theta=\frac{a^2}{b^2}\left(b^2\cos^2\theta+a^2\sin^2\theta\right)$$

Therefore $\frac{PX}{PY}=\frac{\frac{b}{a}\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}{\frac{a}{b}\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{b^2}{a^2}$.

5 Solution

The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. Point X has coordinates $(a\cos\theta, 0)$ and point Y has coordinates $(0, -b\cot\theta)$. Hence

$PX^2=(a\sec\theta-a\cos\theta)^2+b^2\tan^2\theta=a^2\cos^2\theta\tan^4\theta+b^2\tan^2\theta$,

$PY^2=a^2\sec^2\theta+(b\tan\theta+b\cot\theta)^2=a^2\sec^2\theta+b^2\sec^2\theta\operatorname{cosec}^2\theta$.

Therefore $\frac{PX}{PY}=\frac{\sqrt{\sin^2\theta(a^2\tan^2\theta+b^2\sec^2\theta)}}{\sqrt{\operatorname{cosec}^2\theta(a^2\tan^2\theta+b^2\sec^2\theta)}}=\sin^2\theta$. If P is an extremity of a latus rectum, then $a\sec\theta=\pm ae$. Thus $\cos\theta=\pm\frac{1}{e}$. But $\frac{PX}{PY}=1-\cos^2\theta$. Hence $\frac{PX}{PY}=1-\frac{1}{e^2}=\frac{e^2-1}{e^2}$.

6 Solution

The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a,0)$ has equation $x=a$. This tangent meets the asymptote $y=\frac{b}{a}x$ at the point (a,b) and the asymptote $y=-\frac{b}{a}x$ at the point $(a,-b)$. Hence $OT^2=a^2+b^2=a^2e^2=OS^2$. Therefore $OT=OS$.

7 Solution

The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. This tangent meets the asymptote $y=\frac{b}{a}x$ at the point $M\left(a\frac{\cos\theta}{1-\sin\theta}, b\frac{\cos\theta}{1-\sin\theta}\right)$ and meets the asymptote $y=-\frac{b}{a}x$ at the point $N\left(a\frac{\cos\theta}{1+\sin\theta}, -b\frac{\cos\theta}{1+\sin\theta}\right)$. Hence

$$PM^2 = \left(a\sec\theta - a\frac{\cos\theta}{1-\sin\theta}\right)^2 + \left(b\tan\theta - b\frac{\cos\theta}{1-\sin\theta}\right)^2 = a^2\tan^2\theta + b^2\sec^2\theta,$$

$$PN^2 = \left(a\sec\theta - a\frac{\cos\theta}{1+\sin\theta}\right)^2 + \left(b\tan\theta + b\frac{\cos\theta}{1+\sin\theta}\right)^2 = a^2\tan^2\theta + b^2\sec^2\theta.$$

Therefore $PM = PN$.

8 Solution

(a) Let PQ be a chord of contact of tangents from $T(x_0, y_0)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If $T(x_0, y_0)$ lies on the directrix $x = \frac{a}{e}$, then $x_0 = \frac{a}{e}$ and the chord PQ has equation $\frac{x}{ae} - \frac{yy_0}{b^2} = 1$. But $S(ae,0)$ satisfies this equation and hence PQ is a focal chord through S. Similarly, if T lies on $x = -\frac{a}{e}$, then PQ is a focal chord through $S'(-ae,0)$.

(b) Let PQ be a focal chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If tangents at P and Q meet in $T(x_0, y_0)$, then PQ has equation $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. Hence if $S(ae,0)$ lies on PQ, then $x_0 = \frac{a}{e}$ and T lies on the directrix $x = \frac{a}{e}$; if $S'(-ae,0)$ lies on PQ, then $x_0 = -\frac{a}{e}$ and T lies on the directrix $x = -\frac{a}{e}$.

9 Solution

(a) The hyperbola has parametric equations $x = a\sec\theta$ and $y = b\tan\theta$. Hence $\frac{dy}{dx} = \frac{b\sec\theta}{a\tan\theta}$. If $y = mx + k$ is a tangent to the hyperbola at $P(a\sec\phi, b\tan\phi)$, then

$m = \frac{dy}{dx}$ at P $\quad\Rightarrow ma\tan\phi - b\sec\phi = 0$

(1)

P lies on $y = mx + k$ $\quad\Rightarrow ma\sec\phi - b\tan\phi = -k$

(2)

$(2)^2-(1)^2 \Rightarrow m^2a^2(\sec^2\phi-\tan^2\phi)+b^2(\tan^2\phi-\sec^2\phi)=k^2 \Rightarrow m^2a^2-b^2=k^2$.

(b) $(2)\times\sec\phi-(1)\times\tan\phi \Rightarrow ma(\sec^2\phi-\tan^2\phi)=-k\sec\phi \Rightarrow a\sec\phi=-\dfrac{ma^2}{k}$,

$(2)\times\tan\phi-(1)\times\sec\phi \Rightarrow b(\sec^2\phi-\tan^2\phi)=-k\tan\phi \Rightarrow b\tan\phi=-\dfrac{b^2}{k}$.

Therefore the point of contact of the tangent $y=mx+k$ is $P\left(-\dfrac{ma^2}{k},-\dfrac{b^2}{k}\right)$. Now tangents from the point $(1,3)$ to the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{15}=1$ have equations of the form $y-3=m(x-1)$, that is, $y=mx+(3-m)$. Hence

$m^2a^2-b^2=k^2 \Rightarrow 4m^2-15=(3-m)^2 \Rightarrow 3m^2+6m-24=0 \Rightarrow (m-2)(m+4)=0$.

$\therefore m=2,\ k=3-m=1$ and $P\left(-\dfrac{ma^2}{k},-\dfrac{b^2}{k}\right)\equiv P(-8,-15)$,

or $m=-4,\ k=3-m=7$ and $P\left(-\dfrac{ma^2}{k},-\dfrac{b^2}{k}\right)\equiv P\left(\dfrac{16}{7},-\dfrac{15}{7}\right)$.

Hence the tangents from the point $(1,3)$ to the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{15}=1$ are

$y=2x+1$, with point of contact $P(-8,-15)$ and

$y=-4x+7$, with point of contact $P\left(\dfrac{16}{7},-\dfrac{15}{7}\right)$.

10 Solution

The chord of contact of tangents from the point $(1,3)$ to the hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{15}=1$ has equation $\dfrac{x}{4}-\dfrac{3y}{15}=1 \Rightarrow 5x-4y=20$. Let $T(x_0,y_0)$ be a point of contact. Then

T lies on the chord $\Rightarrow 5x_0-4y_0=20$,

T lies on the hyperbola $\Rightarrow \dfrac{x_0^2}{4}-\dfrac{y_0^2}{15}=1$.

Hence $\dfrac{x_0^2}{4}-\dfrac{(5x_0-20)^2}{16\times 15}=1 \Rightarrow 7x_0^2+40x_0-128=0 \Rightarrow (7x_0-16)(x_0+8)=0$

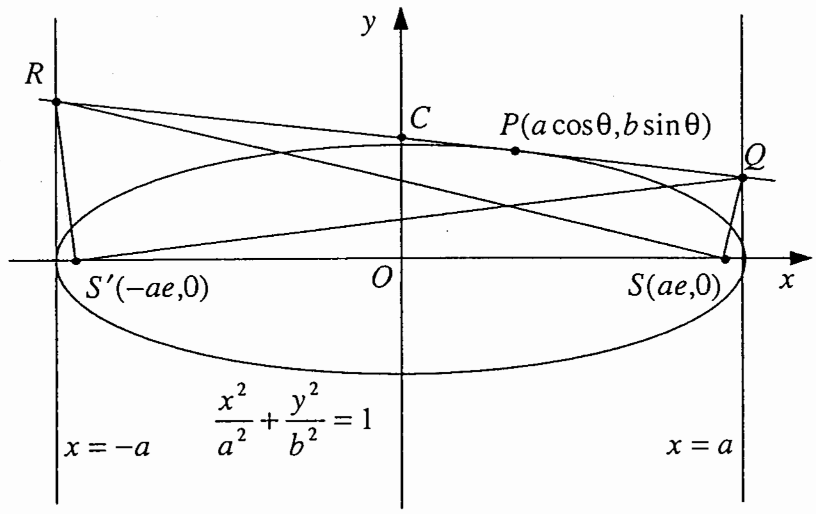
$$\therefore x_0 = \frac{16}{7},\ y_0 = \frac{5x_0 - 20}{4} = -\frac{15}{7} \qquad \text{or} \qquad x_0 = -8,\ y_0 = \frac{5x_0 - 20}{4} = -15.$$

Equation of tangent at the point $T(x_0, y_0)$ is $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. Therefore the tangents from the point (1,3) to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1$ are

$y = -4x + 7$, with point of contact $P\left(\frac{16}{7}, -\frac{15}{7}\right)$ and

$y = 2x + 1$, with point of contact $P(-8, -15)$.

11 Solution



Let the tangent at P meet $x = a$, $x = -a$ in Q, R respectively. Let QR meet the y-axis in C. Tangent PR has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

Hence Q has coordinates $\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$

and R has coordinates $\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$.

$$\text{Gradient } QS \times \text{gradient } RS = \frac{b(1-\cos\theta)}{a(1-e)\sin\theta} \cdot \frac{b(1+\cos\theta)}{-a(1+e)\sin\theta} = -\frac{b^2}{a^2(1-e^2)} \cdot \frac{1-\cos^2\theta}{\sin^2\theta}.$$

Then $b^2 = a^2(1-e^2) \Rightarrow \text{gradient } QS \times \text{gradient } RS = -1 \therefore QS \perp RS$. Similarly, replacing e by $-e$, $QS' \perp RS'$. Hence QR subtends angles of $90°$ at each of S and S', and Q, S, R, S' are concyclic, with QR the diameter of the circle through the points. The y-axis is the perpendicular bisector of the chord SS', hence the centre of this circle is the point C where the diameter QR meets the y-axis.

If $P\left(1,\dfrac{2\sqrt{2}}{3}\right)$ lies on the ellipse $\dfrac{x^2}{9}+y^2=1$, then QR has equation $\dfrac{x}{9}+\dfrac{2\sqrt{2}y}{3}=1$ and meets the y-axis in $C\left(0,\dfrac{3}{2\sqrt{2}}\right)$. Also $b^2=a^2(1-e^2)$ gives $e^2=\dfrac{8}{9}$, and S has coordinates $\left(2\sqrt{2},0\right)$. Hence $CS^2=\dfrac{73}{8}$ and the circle through Q, S, R, S' has equation $x^2+\left(y-\dfrac{3}{2\sqrt{2}}\right)^2=\dfrac{73}{8}$.

Exercise 3.4

1 Solution

(a) For the hyperbola $xy = 8$ we have $c^2 = 8 \Rightarrow c = 2\sqrt{2}$. Hence the hyperbola $xy = 8$ has

eccentricity $e = \sqrt{2}$,

foci $S(c\sqrt{2}, c\sqrt{2}) = S(4,4)$

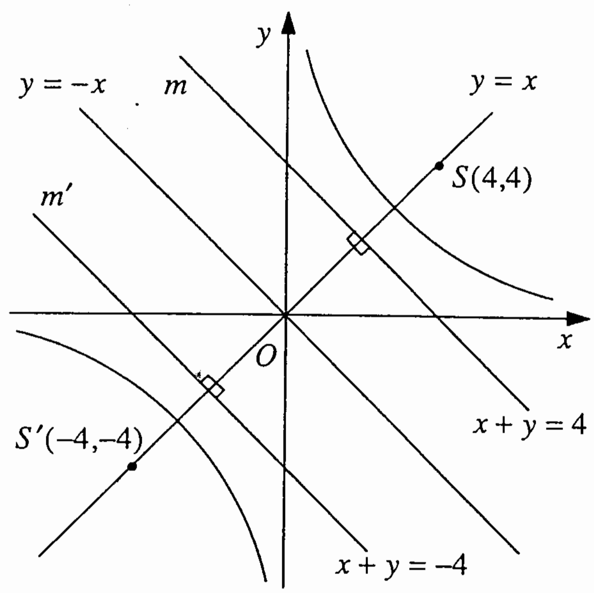
and

$S'(-c\sqrt{2}, -c\sqrt{2}) = S(-4,-4)$,

directrices

$x + y = \pm c\sqrt{2} \Rightarrow x + y = \pm 4$,

asymptotes $x = 0$ and $y = 0$.



(b) For the hyperbola $xy = 16$ we have $c^2 = 16 \Rightarrow c = 4$. Hence the hyperbola $xy = 16$ has

eccentricity $e = \sqrt{2}$,

foci

$S(c\sqrt{2}, c\sqrt{2}) = S(4\sqrt{2}, 4\sqrt{2})$

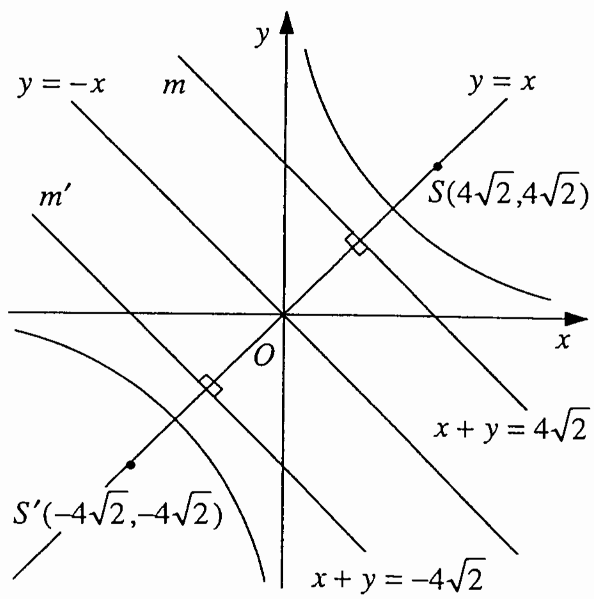
and

$S'(-c\sqrt{2}, -c\sqrt{2}) = S(-4\sqrt{2}, -4\sqrt{2})$,

directrices

$x + y = \pm c\sqrt{2} \Rightarrow x + y = \pm 4\sqrt{2}$,

asymptotes $x = 0$ and $y = 0$.



2 Solution

(a) For the hyperbola $xy = 4$ we have $c^2 = 4 \Rightarrow c = 2$. Hence the hyperbola $xy = 4$

has parametric equations $x = ct,\ y = \frac{c}{t} \Rightarrow x = 2t,\ y = \frac{2}{t}$.

(b) For the hyperbola $xy = 25$ we have $c^2 = 25 \Rightarrow c = 5$. Hence the hyperbola $xy = 25$ has parametric equations $x = ct,\ y = \frac{c}{t} \Rightarrow x = 5t,\ y = \frac{5}{t}$.

3 Solution

(a) The hyperbola $x = 4t,\ y = \frac{4}{t}$ has Cartesian equation $xy = 4t \cdot \frac{4}{t} \Rightarrow xy = 16$.

(b) The hyperbola $x = 3t,\ y = \frac{3}{t}$ has Cartesian equation $xy = 3t \cdot \frac{3}{t} \Rightarrow xy = 9$.

4 Solution

(a) For the hyperbola $xy = 8$ we have $c^2 = 8$. Hence the tangent to the hyperbola $xy = 8$ at the point $P(x_1, y_1) = P(4,2)$ has equation $xy_1 + yx_1 = 2c^2 \Rightarrow x + 2y = 8$ and the normal has equation $xx_1 - yy_1 = x_1^2 - y_1^2 \Rightarrow 2x - y = 6$.

(b) For the hyperbola $xy = 12$ we have $c^2 = 12$. Hence the tangent to the hyperbola $xy = 12$ at the point $P(x_1, y_1) = P(-3,-4)$ has equation $xy_1 + yx_1 = 2c^2 \Rightarrow 4x + 3y = -24$ and the normal has equation $xx_1 - yy_1 = x_1^2 - y_1^2 \Rightarrow 3x - 4y = 7$.

(c) For the hyperbola $x = 2t,\ y = \frac{2}{t}$ we have $c = 2$. Hence the tangent to the hyperbola $x = 2t,\ y = \frac{2}{t}$ at the point where $t = 4$ has equation $x + t^2 y = 2ct \Rightarrow x + 16y = 16$ and the normal has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right) \Rightarrow 32x - 2y = 255$.

(d) For the hyperbola $x = 3t,\ y = \frac{3}{t}$ we have $c = 3$. Hence the tangent to the hyperbola $x = 3t,\ y = \frac{3}{t}$ at the point where $t = -1$ has equation $x + t^2 y = 2ct \Rightarrow x + y = -6$ and the normal has equation

$$tx-\frac{y}{t}=c\left(t^2-\frac{1}{t^2}\right)\Rightarrow x-y=0.$$

5 Solution

(a) For the hyperbola $xy=10$ we have $c^2=10$. Hence the chord of contact of tangents from the point $T(x_0,y_0)=T(2,1)$ to the hyperbola $xy=10$ has equation $xy_0+yx_0=2c^2\Rightarrow x+2y=20$.

(b) For the hyperbola $xy=6$ we have $c^2=6$. Hence the chord of contact of tangents from the point $T(x_0,y_0)=T(1,-2)$ to the hyperbola $xy=6$ has equation $xy_0+yx_0=2c^2\Rightarrow 2x-y=-12$.

6 Solution

(a) The hyperbola $xy=c^2$ has parametric equations $x=ct$ and $y=\frac{c}{t}$. Hence $\frac{dy}{dx}=-\frac{1}{t^2}$. If $y=mx+k$ is a tangent to the hyperbola at $P\left(cp,\frac{c}{p}\right)$, then

$m=\frac{dy}{dx}$ at P $\qquad\Rightarrow mp^2+1=0$

(1)

P lies on $y=mx+k$ $\qquad\Rightarrow mcp-\frac{c}{p}=-k$

(2)

$\therefore (1)\Rightarrow p^2=-\frac{1}{m}$. Thus $(2)^2\Rightarrow m^2c^2p^2-2mc^2+\frac{c^2}{p^2}=k^2\Rightarrow 4mc^2+k^2=0$.

(b) $(1)\times\frac{c}{p}+(2)\Rightarrow 2mcp=-k\Rightarrow cp=-\frac{k}{2m}$,

$(1)\times\frac{c}{p}-(2)\Rightarrow\frac{2c}{p}=k\Rightarrow\frac{c}{p}=\frac{k}{2}$.

Therefore the point of contact of the tangent $y=mx+k$ is $P\left(-\frac{k}{2m},\frac{k}{2}\right)$. Now tangents from the point $(-1,-3)$ to the hyperbola $xy=4$ have equations of the form

$y+3=m(x+1)$, that is, $y=mx+(m-3)$. Hence

$4mc^2+k^2=0 \Rightarrow 16m+(m-3)^2=0 \Rightarrow m^2+10m+9=0 \Rightarrow (m+1)(m+9)=0$.

$\therefore m=-1,\ k=m-3=-4$ and $P\left(-\dfrac{k}{2m},\dfrac{k}{2}\right) \equiv P(-2,-2)$,

or $m=-9,\ k=m-3=-12$ and $P\left(-\dfrac{k}{2m},\dfrac{k}{2}\right) \equiv P\left(-\dfrac{2}{3},-6\right)$.

Hence the tangents from the point $(-1,-3)$ to the hyperbola $xy=4$ are

$y=-x-4$, with point of contact $P(-2,-2)$ and

$y=-9x-12$, with point of contact $P\left(-\dfrac{2}{3},-6\right)$.

7 Solution

The chord of contact of tangents from the point $(-1,-3)$ to the hyperbola $xy=4$ has equation $3x+y=-8$. Let $T(x_0,y_0)$ be a point of contact. Then

T lies on the chord $\qquad \Rightarrow 3x_0+y_0=-8$,

T lies on the hyperbola $\qquad \Rightarrow x_0y_0=4$.

Hence $x_0(-8-3x_0)=4 \Rightarrow 3x_0^2+8x_0+4=0 \Rightarrow (3x+2)(x+2)=0$

$\therefore x_0=-\dfrac{2}{3},\ y_0=-8-3x_0=-6 \qquad$ or $\qquad x_0=-2,\ y_0=-8-3x_0=-2$.

Equation of tangent at the point $T(x_0,y_0)$ is $xy_0+yx_0=2c^2$. Therefore the tangents from the point $(-1,-3)$ to the hyperbola $xy=4$ are

$y=-x-4$, with point of contact $P(-2,-2)$ and

$y=-9x-12$, with point of contact $P\left(-\dfrac{2}{3},-6\right)$.

8 Solution

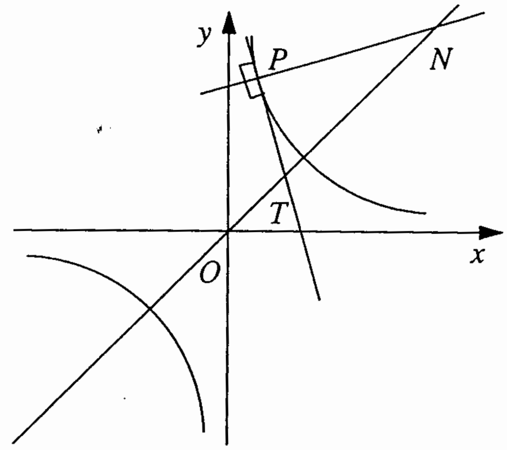
The gradient of PR is $\dfrac{c\left(\dfrac{1}{p}-\dfrac{1}{r}\right)}{c(p-r)}=-\dfrac{1}{pr}$, the gradient of QR is $\dfrac{c\left(\dfrac{1}{q}-\dfrac{1}{r}\right)}{c(q-r)}=-\dfrac{1}{qr}$.

Therefore $\qquad PR \perp QR \Rightarrow \text{gradient } PR \times \text{gradient } QR=-1 \Rightarrow \dfrac{1}{pqr^2}=-1 \Rightarrow r^2=-\dfrac{1}{pq}$.

The normal at the point $R\left(cr, \frac{c}{r}\right)$ has gradient r^2, the gradient of PQ is $\frac{c\left(\frac{1}{p}-\frac{1}{q}\right)}{c(p-q)} = -\frac{1}{pq}$. Since $r^2 = -\frac{1}{pq}$, then gradient of the normal at R equals to gradient of PQ. Thus the normal at the point R is parallel to the chord PQ.

9 Solution

The tangent to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. Hence the point T has coordinates $\left(\frac{2ct}{1+t^2}, \frac{2ct}{1+t^2}\right)$. The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$



has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. Therefore the point N has coordinates $\left(c\frac{t^2+1}{t}, c\frac{t^2+1}{t}\right)$.

(a)

$$OP^2 = c^2t^2 + \frac{c^2}{t^2}, \quad PN^2 = \left(ct - c\frac{t^2+1}{t}\right)^2 + \left(\frac{c}{t} - c\frac{t^2+1}{t}\right)^2 = \frac{c^2}{t^2} + c^2t^2 \Rightarrow OP = PN$$

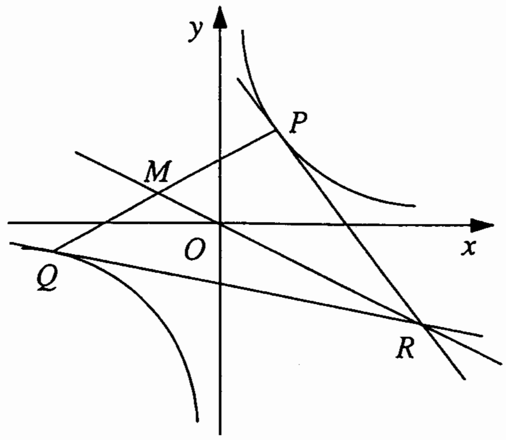
(b) $OT = \frac{2ct}{1+t^2}\sqrt{2}, \quad ON = c\frac{t^2+1}{t}\sqrt{2} \Rightarrow OT \times ON = 4c^2$.

10 Solution

Since $R(x_0, y_0)$ lies on the tangent at the point $P\left(cp, \frac{c}{p}\right)$, then $x_0 + p^2 y_0 = 2cp$.

Since $R(x_0, y_0)$ lies on the tangent at the point $Q\left(cq, \frac{c}{q}\right)$, then $x_0 + q^2 y_0 = 2cq$.

Thus $x_0 = \frac{2cpq}{p+q}$ and $y_0 = \frac{2c}{p+q}$. Then



OR has equation $y = \frac{y_0}{x_0} x = \frac{x}{pq}$. The point $M(x_1, y_1)$ lies on OR. Therefore $y_1 = \frac{x_1}{pq}$. Since PQ is the chord of contact of tangents from the point $R(x_0, y_0)$, then PQ has equation $xy_0 + yx_0 = 2c^2$. $M(x_1, y_1)$ lies on PQ. Hence $\frac{x_1}{pq} + y_1 = c\frac{p+q}{pq}$.

Thus $y_1 = \frac{1}{2}\left(\frac{c}{p} + \frac{c}{q}\right)$ and $x_1 = \frac{1}{2}(cp + cq)$. Therefore M is the midpoint of PQ.

11 Solution

Let R has coordinates (x_0, y_0). PQ is the chord of contact of tangents from R to the hyperbola $xy = 9$. Hence PQ has equation $xy_0 + yx_0 = 18$. Then (6,2) lies on PQ. Therefore $x_0 + 3y_0 = 9$. Thus the locus of R has equation $x + 3y = 9$.

12 Solution

Let R has coordinates (x_0, y_0). PQ is the chord of contact of tangents from R to the hyperbola $xy = c^2$. Hence PQ has equation $xy_0 + yx_0 = 2c^2$. Then $(a,0)$ lies on PQ. Therefore $ay_0 = 2c^2$. Thus the locus of R has equation $y = \frac{2c^2}{a}$.

Diagnostic test 3

1 Solution

For the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$

we have

$a=2,\ b=\sqrt{3}\Rightarrow b<a$
$b^2=a^2(1-e^2)$,

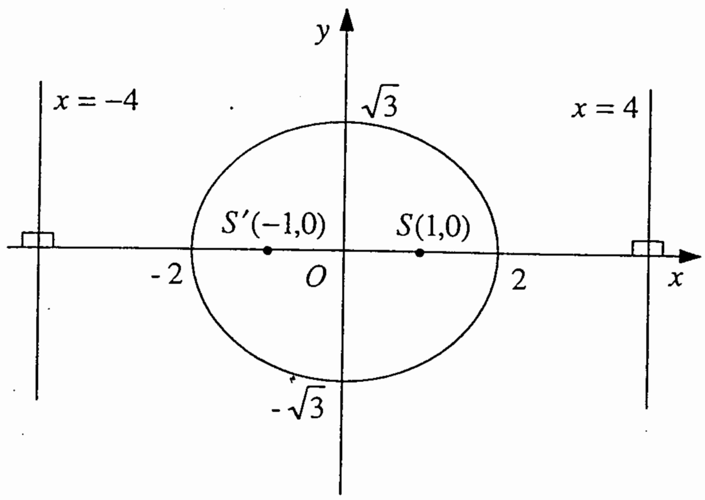
eccentricity:

$e=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$,

foci:

$(\pm ae,0)\Rightarrow(\pm 1,0)$,

directrices: $x=\pm\frac{a}{e}\Rightarrow x=\pm 4$.



2 Solution

For the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$ we have

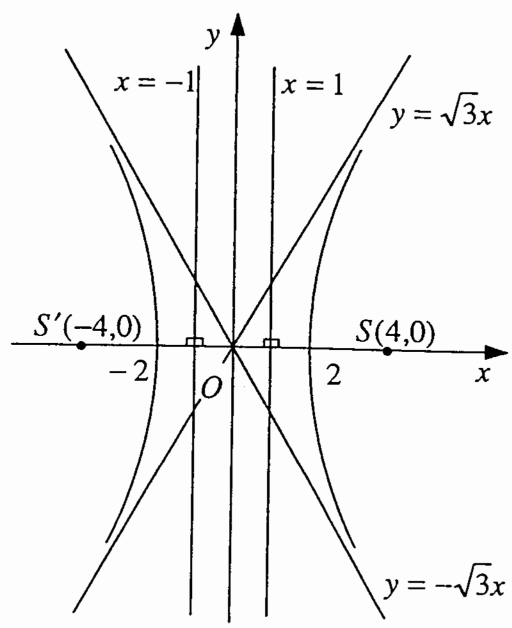
$a=2,\ b=2\sqrt{3}$
$b^2=a^2(e^2-1)$,

eccentricity: $e=\sqrt{1+\frac{12}{4}}=2$,

foci: $(\pm ae,0)\Rightarrow(\pm 4,0)$,

directrices: $x=\pm\frac{a}{e}\Rightarrow x=\pm 1$,

asymptotes: $y=\pm\frac{b}{a}x\Rightarrow y=\pm\sqrt{3}x$.



3 Solution

If P lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the foci $S(ae,0)$ and $S'(-ae,0)$, then

$PS + PS' = 2a$. For the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$ we have $a = 3$. Hence if $PS = 2$, then $PS' = 6 - 2 = 4$.

4 Solution

Since foci of a hyperbola are on the x-axes, then the equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Thus we need to find the parameters a and b. Coordinates of the foci are $(\pm ae, 0)$. Therefore the distance between the foci is $2ae = 16$. The equations of the directrices are $x = \pm\frac{a}{e}$. Hence the distance between the directrices is $2 \cdot \frac{a}{e} = 4$. Thus we have two equations $ae = 8$ and $\frac{a}{e} = 2$. From the first equation we get $e = \frac{8}{a}$. Substituting the expression for the e to the second equation we obtain $a^2 = 16$. Therefore $a = 4$ and $e = \frac{8}{4} = 2$. Then $b^2 = a^2(e^2 - 1) = 16 \cdot (4 - 1) = 48$. Hence the Cartesian equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{48} = 1$.

5 Solution

(a) Cartesian equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Hence $a = 3$ and $b = 2$. Therefore the ellipse has parametric equations $x = 3\cos\theta$ and $y = 2\sin\theta$, $-\pi < \theta \le \pi$.

(b) Cartesian equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Hence $a = 3$ and $b = 4$. Therefore the hyperbola has parametric equations $x = 3\sec\theta$ and $y = 4\tan\theta$, $-\pi < \theta \le \pi$, $\theta \ne \pm\frac{\pi}{2}$.

6 Solution

(a) The ellipse has parametric equations $x = 4\cos\theta$, $y = 3\sin\theta$. Therefore

$\frac{x^2}{16}+\frac{y^2}{9}=\cos^2\theta+\sin^2\theta=1$. Hence the Cartesian equation of the ellipse is

$\frac{x^2}{16}+\frac{y^2}{9}=1$.

(b) The hyperbola has parametric equations $x=4\sec\theta$, $y=5\tan\theta$. Therefore

$\frac{x^2}{16}-\frac{y^2}{25}=\sec^2\theta-\tan^2\theta=1$. Hence the Cartesian equation of the hyperbola is

$\frac{x^2}{16}-\frac{y^2}{25}=1$.

7 Solution

(a) Chord PQ has equation $x=ae$, P has coordinates $(a\sec\theta,b\tan\theta)$. Hence $a\sec\theta=ae$. Thus $\sec\theta=e$.

(b) Length of the chord PQ is

$|b\tan\theta-b\tan(-\theta)|=2b|\tan\theta|=2b\sqrt{\sec^2\theta-1}=2b\sqrt{e^2-1}$. But for the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ we have $b^2=a^2(e^2-1)$. Therefore the length of the chord PQ is

$2b\cdot\frac{b}{a}=\frac{2b^2}{a}$.

8 Solution

(a) Length of PS is $\sqrt{(a\cos\theta-ae)^2+(b\sin\theta)^2}=\sqrt{a^2(\cos\theta-e)^2+b^2\sin^2\theta}$. For the

ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we have $b^2=a^2(1-e^2)$. Therefore the length of PS is

$\sqrt{a^2(\cos\theta-e)^2+a^2(1-e^2)\sin^2\theta}=a\sqrt{\cos^2\theta-2e\cos\theta+e^2+\sin^2\theta-e^2\sin^2\theta}=$
$a\sqrt{(\cos^2\theta+\sin^2\theta)-2e\cos\theta+e^2(1-\sin^2\theta)}=a\sqrt{1-2e\cos\theta+e^2\cos^2\theta}=a\sqrt{(1-e\cos\theta)^2}$

Hence the length of PS is $a(1-e\cos\theta)$.

Length of PS' is $\sqrt{(a\cos\theta+ae)^2+(b\sin\theta)^2}=\sqrt{a^2(\cos\theta+e)^2+b^2\sin^2\theta}$. For the

ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we have $b^2=a^2(1-e^2)$. Therefore the length of PS' is

$$\sqrt{a^2(\cos\theta+e)^2+a^2(1-e^2)\sin^2\theta}=a\sqrt{\cos^2\theta+2e\cos\theta+e^2+\sin^2\theta-e^2\sin^2\theta}=$$
$$a\sqrt{(\cos^2\theta+\sin^2\theta)+2e\cos\theta+e^2(1-\sin^2\theta)}=a\sqrt{1+2e\cos\theta+e^2\cos^2\theta}=a\sqrt{(1+e\cos\theta)^2}$$

Hence the length of PS' is $a(1+e\cos\theta)$.

(b) $PS+PS'=a(1-e\cos\theta)+a(1+e\cos\theta)=2a$.

9 Solution

(a) The tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$ at the point $(2,1)$ has equation $\frac{2x}{8}+\frac{y}{2}=1\Rightarrow x+2y=4$. The normal to the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$ at the point $(2,1)$ has equation $\frac{8x}{2}-\frac{2y}{1}=8-2\Rightarrow 2x-y=3$.

(b) The tangent to the ellipse $x=4\cos\theta,\ y=2\sin\theta$ at the point where $\theta=\frac{\pi}{3}$ has equation $\frac{x\cos\frac{\pi}{3}}{4}+\frac{y\sin\frac{\pi}{3}}{2}=1\Rightarrow x+2\sqrt{3}y=8$. The normal to the ellipse $x=4\cos\theta,\ y=2\sin\theta$ at the point where $\theta=\frac{\pi}{3}$ has equation $\frac{4x}{\cos\frac{\pi}{3}}-\frac{2y}{\sin\frac{\pi}{3}}=16-4\Rightarrow 6x-\sqrt{3}y=9$.

(c) The tangent to the hyperbola $\frac{x^2}{12}-\frac{y^2}{27}=1$ at the point $(4,3)$ has equation $\frac{4x}{12}-\frac{3y}{27}=1\Rightarrow 3x-y=9$. The normal to the hyperbola $\frac{x^2}{12}-\frac{y^2}{27}=1$ at the point $(4,3)$ has equation $\frac{12x}{4}+\frac{27y}{3}=12+27\Rightarrow x+3y=13$.

(d) The tangent to the hyperbola $x=3\sec\theta,\ y=6\tan\theta$ at the point where $\theta=\frac{\pi}{6}$ has equation $\frac{x\sec\frac{\pi}{6}}{3}-\frac{y\tan\frac{\pi}{6}}{6}=1\Rightarrow 4x-y=6\sqrt{3}$. The normal to the hyperbola

$x = 3\sec\theta,\ y = 6\tan\theta$ at the point where $\theta = \frac{\pi}{6}$ has equation

$$\frac{3x}{\sec\frac{\pi}{6}} + \frac{6y}{\tan\frac{\pi}{6}} = 9 + 36 \Rightarrow x + 4y = 10\sqrt{3}.$$

10 Solution

(a) The chord of contact of tangents from the point $(4,3)$ to the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ has equation $\frac{4x}{8} + \frac{3y}{2} = 1 \Rightarrow x + 3y = 2$.

(b) The chord of contact of tangents from the point $(2,1)$ to the hyperbola $\frac{x^2}{12} - \frac{y^2}{27} = 1$ has equation $\frac{2x}{12} - \frac{y}{27} = 1 \Rightarrow 9x - 2y = 54$.

11 Solution

The tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$. Point X has coordinates $(a\sec\theta, 0)$ and point Y has coordinates $(0, b\operatorname{cosec}\theta)$. Hence

$PX^2 = (a\cos\theta - a\sec\theta)^2 + b^2\sin^2\theta = a^2\sin^2\theta\tan^2\theta + b^2\sin^2\theta$,

$PY^2 = a^2\cos^2\theta + (b\sin\theta - b\operatorname{cosec}\theta)^2 = a^2\cos^2\theta + b^2\cos^2\theta\cot^2\theta$.

Therefore $\frac{PX}{PY} = \frac{\sqrt{\tan^2\theta(a^2\sin^2\theta + b^2\cos^2\theta)}}{\sqrt{\cot^2\theta(a^2\sin^2\theta + b^2\cos^2\theta)}} = \tan^2\theta$. If P is an extremity of a latus rectum, then $a\cos\theta = \pm ae$. Thus $\cos\theta = \pm e$. Hence $\frac{PX}{PY} = \frac{1-e^2}{e^2}$.

12 Solution

The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$. Point X has coordinates $\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$ and point Y has coordinates $\left(0,\dfrac{a^2+b^2}{b}\tan\theta\right)$. Hence

$$PX^2=\left(a-\frac{a^2+b^2}{a}\right)^2\sec^2\theta+b^2\tan^2\theta=\frac{b^4}{a^2}\sec^2\theta+b^2\tan^2\theta=\frac{b^2}{a^2}\left(b^2\sec^2\theta+a^2\tan^2\theta\right)$$

,

$$PY^2=a^2\sec^2\theta+\left(b-\frac{a^2+b^2}{b}\right)^2\tan^2\theta=a^2\sec^2\theta+\frac{a^4}{b^2}\tan^2\theta=\frac{a^2}{b^2}\left(b^2\sec^2\theta+a^2\tan^2\theta\right)$$

.

Therefore $\dfrac{PX}{PY}=\dfrac{\frac{b}{a}\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}{\frac{a}{b}\sqrt{b^2\sec^2\theta+a^2\tan^2\theta}}=\dfrac{b^2}{a^2}$.

13 Solution

For the hyperbola $xy=18$ we have $c^2=18\Rightarrow c=3\sqrt{2}$. Hence the hyperbola $xy=18$ has

eccentricity $e=\sqrt{2}$,

foci $S(c\sqrt{2},c\sqrt{2})=S(6,6)$

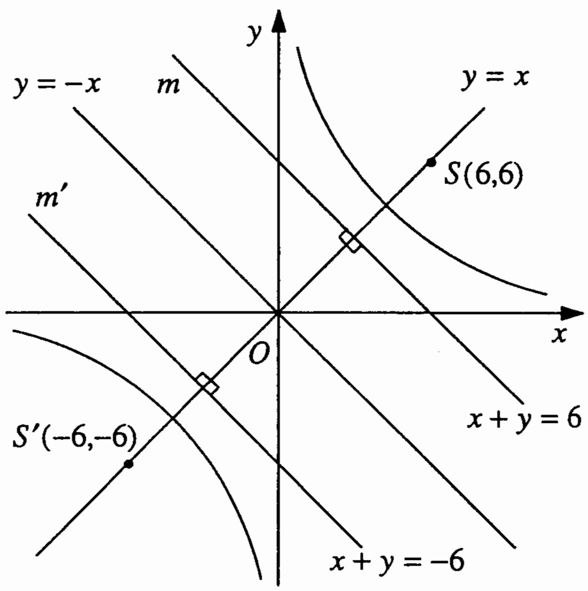
and

$S'(-c\sqrt{2},-c\sqrt{2})=S(-6,-6)$,

directrices

$x+y=\pm c\sqrt{2}\Rightarrow x+y=\pm 6$,

asymptotes $x=0$ and $y=0$.



14 Solution

(a) For the hyperbola $xy=9$ we have $c^2=9\Rightarrow c=3$. Hence the hyperbola $xy=9$ has parametric equations $x=ct$, $y=\dfrac{c}{t}\Rightarrow x=3t$, $y=\dfrac{3}{t}$.

(b) The hyperbola $x=5t$, $y=\dfrac{5}{t}$ has Cartesian equation $xy=5t\cdot\dfrac{5}{t}\Rightarrow xy=25$.

15 Solution

(a) For the hyperbola $xy = 6$ we have $c^2 = 6$. Hence the tangent to the hyperbola $xy = 6$ at the point $P(x_1, y_1) = P(3,2)$ has equation $xy_1 + yx_1 = 2c^2 \Rightarrow 2x + 3y = 12$ and the normal has equation $xx_1 - yy_1 = x_1^2 - y_1^2 \Rightarrow 3x - 2y = 5$.

(b) For the hyperbola $x = 4t,\ y = \dfrac{4}{t}$ we have $c = 4$. Hence the tangent to the hyperbola $x = 4t,\ y = \dfrac{4}{t}$ at the point where $t = 2$ has equation $x + t^2 y = 2ct \Rightarrow x + 4y = 16$ and the normal has equation $tx - \dfrac{y}{t} = c\left(t^2 - \dfrac{1}{t^2}\right) \Rightarrow 4x - y = 30$.

(c) For the hyperbola $xy = 4$ we have $c^2 = 4$. Hence the chord of contact of tangents from the point $T(x_0, y_0) = T(2,-1)$ to the hyperbola $xy = 4$ has equation $xy_0 + yx_0 = 2c^2 \Rightarrow -x + 2y = 8$.

16 Solution

The tangent to the hyperbola $xy = c^2$ at the point $P\left(ct, \dfrac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. Hence the point X has coordinates $(2ct, 0)$ and the point Y has coordinates $\left(0, \dfrac{2c}{t}\right)$.

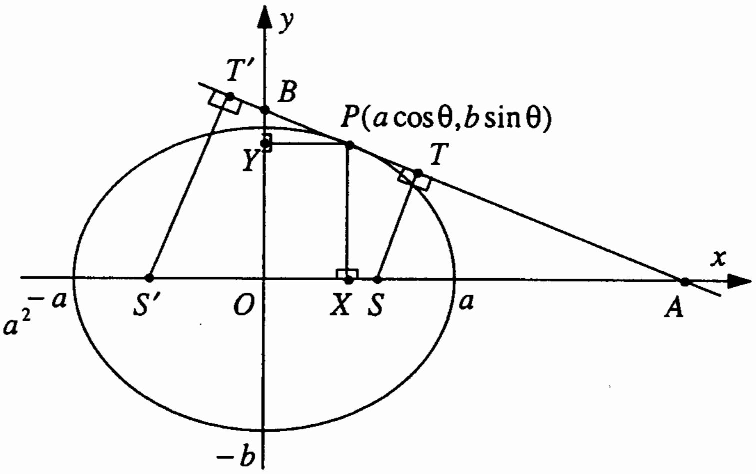
(a) $PX^2 = (ct - 2ct)^2 + \left(\dfrac{c}{t}\right)^2 = c^2\left(t^2 + \dfrac{1}{t^2}\right)$ and $PX^2 = (ct)^2 + \left(\dfrac{c}{t} - \dfrac{2c}{t}\right)^2 = c^2\left(t^2 + \dfrac{1}{t^2}\right)$. Therefore $PX = PY$.

(b) The area of ΔYOX is $\dfrac{1}{2} \cdot OX \cdot OY = \dfrac{1}{2} \cdot 2ct \cdot \dfrac{2c}{t} = 2c^2$. Thus the area of ΔYOX is independent of t.

Further questions 3

1 Solution

The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.



(a) The point A has coordinates $(a\sec\theta, 0)$ and the point X has coordinates $(a\cos\theta, 0)$. Hence $OX\cdot OA = a\cos\theta\cdot a\sec\theta = a^2$.

(b) The point B has coordinates $(0, b\operatorname{cosec}\theta)$ and the point Y has coordinates $(0, b\sin\theta)$. Hence $OY\cdot OB = b\sin\theta\cdot b\operatorname{cosec}\theta = b^2$.

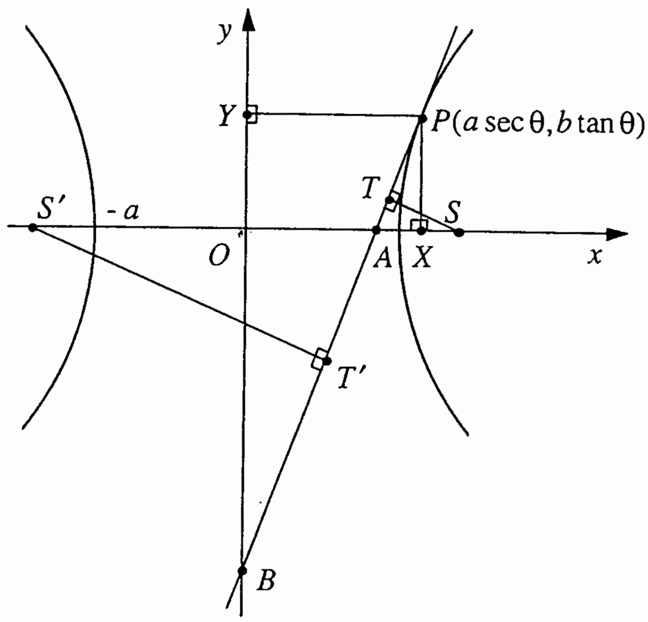
(c) Since S has coordinates $(ae, 0)$, then $ST=\frac{|e\cos\theta-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$. Since S' has coordinates $(-ae, 0)$, then $S'T'=\frac{|-e\cos\theta-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$. Therefore

$ST\cdot S'T'=\frac{1-e^2\cos^2\theta}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}$. But for the ellipse $b^2=a^2(1-e^2)\Rightarrow e^2=1-\frac{b^2}{a^2}$. Hence

$$ST\cdot S'T'=\frac{1-\cos^2\theta+\frac{b^2}{a^2}\cos^2\theta}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}=b^2.$$

2 Solution

The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.



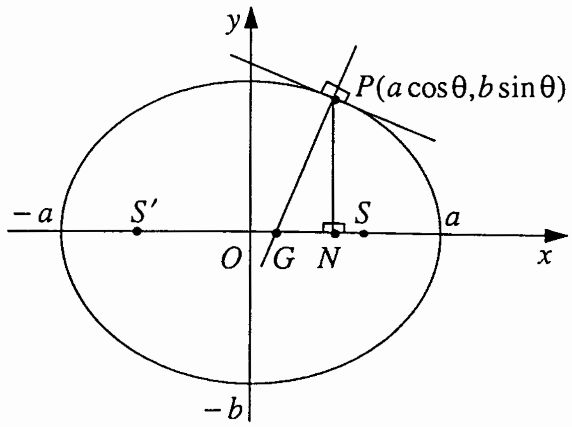
(a) The point A has coordinates $(a\cos\theta, 0)$ and the point X has coordinates $(a\sec\theta, 0)$. Hence $OX\cdot OA=a\sec\theta\cdot a\cos\theta=a^2$.

(b) The point B has coordinates $(0, -b\cot\theta)$ and the point Y has coordinates $(0, b\tan\theta)$. Hence $OY\cdot OB=b\tan\theta\cdot b\cot\theta=b^2$.

(c) Since S has coordinates $(ae, 0)$, then $ST=\dfrac{|e\sec\theta-1|}{\sqrt{\dfrac{\sec^2\theta}{a^2}+\dfrac{\tan^2\theta}{b^2}}}$. Since S' has coordinates $(-ae, 0)$, then $S'T'=\dfrac{|-e\sec\theta-1|}{\sqrt{\dfrac{\sec^2\theta}{a^2}+\dfrac{\tan^2\theta}{b^2}}}$. Hence $ST\cdot S'T'=\dfrac{e^2\sec^2\theta-1}{\dfrac{\sec^2\theta}{a^2}+\dfrac{\tan^2\theta}{b^2}}$. But for the hyperbola

$b^2=a^2(e^2-1)\Rightarrow e^2=\dfrac{b^2}{a^2}+1$. Thus $ST\cdot S'T'=\dfrac{\dfrac{b^2}{a^2}\sec^2\theta+\sec^2\theta-1}{\dfrac{\sec^2\theta}{a^2}+\dfrac{\tan^2\theta}{b^2}}=b^2$.

3 Solution



The normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. The point G has coordinates $\left(\frac{a^2-b^2}{a}\cos\theta, 0\right)$.

(a) The point N has coordinates $(a\cos\theta, 0)$. Therefore $OG=\frac{a^2-b^2}{a^2}ON$. But for the ellipse $b^2=a^2(1-e^2)\Rightarrow\frac{a^2-b^2}{a^2}=e^2$. Thus $OG=e^2ON$.

(b) Since the focus S has coordinates $(ae, 0)$,

then $SG=\left|ae-\frac{a^2-b^2}{a}\cos\theta\right|=ae(1-e\cos\theta)$

and $SP=\sqrt{(ae-a\cos\theta)^2+b^2\sin^2\theta}=a\sqrt{(e-\cos\theta)^2+(1-e^2)\sin^2\theta}$

$=a\sqrt{1-2e\cos\theta+e^2\cos^2\theta}=a(1-e\cos\theta)$.

Hence $SG=eSP$. Since the focus S' has coordinates $(-ae, 0)$,

then $S'G=\left|-ae-\frac{a^2-b^2}{a}\cos\theta\right|=ae(1+e\cos\theta)$

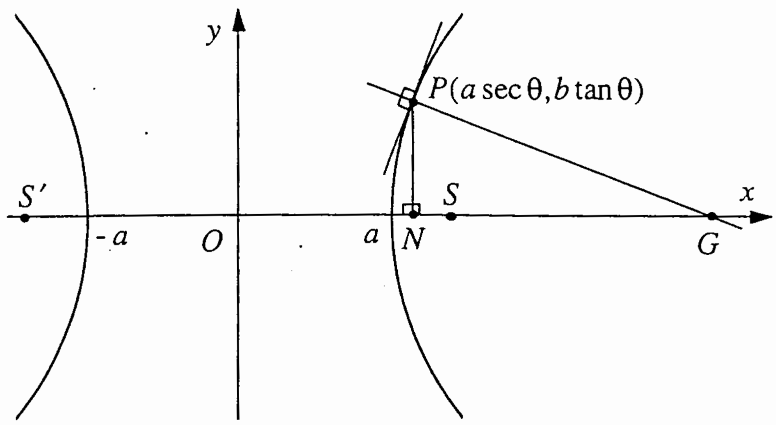
and $S'P=\sqrt{(-ae-a\cos\theta)^2+b^2\sin^2\theta}=a\sqrt{(e+\cos\theta)^2+(1-e^2)\sin^2\theta}$

$=a\sqrt{1+2e\cos\theta+e^2\cos^2\theta}=a(1+e\cos\theta)$.

Hence $S'G=eS'P$.

4 Solution

The normal to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\dfrac{ax}{\sec\theta}+\dfrac{by}{\tan\theta}=a^2+b^2$. The point G has coordinates $\left(\dfrac{a^2+b^2}{a}\sec\theta, 0\right)$.



(a) The point N has coordinates $(a\sec\theta, 0)$. Therefore $OG=\dfrac{a^2+b^2}{a^2}ON$. But for the hyperbola $b^2=a^2(e^2-1)\Rightarrow\dfrac{a^2+b^2}{a^2}=e^2$. Thus $OG=e^2ON$.

(b) Since the focus S has coordinates $(ae,0)$,

then $SG=\left|ae-\dfrac{a^2+b^2}{a}\sec\theta\right|=ae|1-e\sec\theta|$

and $SP=\sqrt{(ae-a\sec\theta)^2+b^2\tan^2\theta}=a\sqrt{(e-\sec\theta)^2+(e^2-1)\tan^2\theta}$

$$=a\sqrt{1-2e\sec\theta+e^2\sec^2\theta}=a|1-e\sec\theta|.$$

Hence $SG=eSP$. Since the focus S' has coordinates $(-ae,0)$,

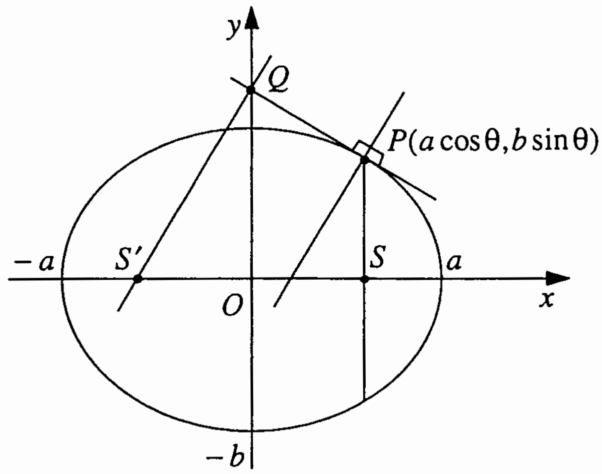
then $S'G=\left|-ae-\dfrac{a^2+b^2}{a}\sec\theta\right|=ae|1+e\sec\theta|$

and $S'P=\sqrt{(-ae-a\sec\theta)^2+b^2\tan^2\theta}=a\sqrt{(e+\sec\theta)^2+(e^2-1)\tan^2\theta}$

$$=a\sqrt{1+2e\sec\theta+e^2\sec^2\theta}=a|1+e\sec\theta|.$$

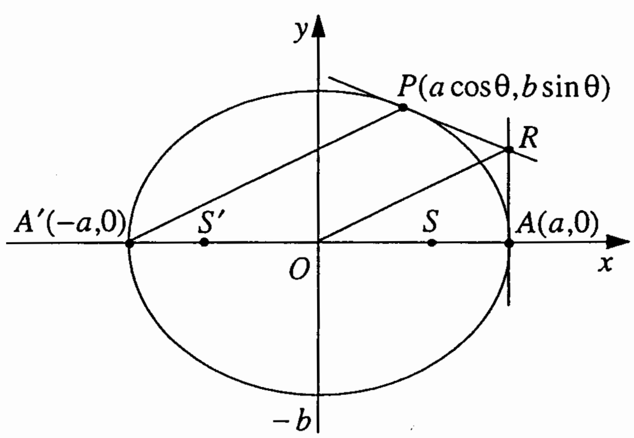
Hence $S'G=eS'P$.

5 Solution



The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. Hence the point Q has coordinates $(0, b\operatorname{cosec}\theta)$. Thus the gradient of QS' is $\frac{b\operatorname{cosec}\theta}{ae}$. The gradient of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ is $\frac{a\sin\theta}{b\cos\theta}$. Since P lies at an extremity of a latus rectum through the focus $S(ae,0)$, then $\cos\theta = e$ and $\sin\theta = \sqrt{1-e^2} = \frac{b}{a}$. Therefore the gradient of QS' is $\frac{b}{ae}\cdot\frac{a}{b}=\frac{1}{e}$ and the gradient of the normal at P is $\frac{a}{be}\cdot\frac{b}{a}=\frac{1}{e}$. Hence the normal at P is parallel to QS'.

6 Solution



The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. Hence the point R has coordinates $\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$. Thus the gradient of OR is $\frac{b(1-\cos\theta)}{a\sin\theta}$. The gradient of $A'P$ is $\frac{b\sin\theta}{a(\cos\theta+1)}=\frac{b\sin\theta(1-\cos\theta)}{a(\cos\theta+1)(1-\cos\theta)}=\frac{b\sin\theta(1-\cos\theta)}{a(1-\cos^2\theta)}=\frac{b(1-\cos\theta)}{a\sin\theta}$. Therefore OR

is parallel to $A'P$.

7 Solution

The tangent to the ellipse $x^2+2y^2=19$ at the point $P(x_0,y_0)$ has equation $xx_0+2yy_0=19$. If this tangent is parallel to $x+6y=5$, then $\frac{2y_0}{x_0}=6 \Rightarrow y_0=3x_0$.

Since the point $P(x_0,y_0)$ lies on the ellipse, then $x_0^2+2y_0^2=19$. Therefore $x_0^2+2\cdot 9x_0^2=19 \Rightarrow x_0^2=1$. Hence the tangents to the ellipse $x^2+2y^2=19$ are

$x+6y=19$, with point of contact $P(1,3)$ and

$x+6y=-19$, with point of contact $P(-1,-3)$.

8 Solution

The tangent to the hyperbola $2x^2-3y^2=5$ at the point $P(x_0,y_0)$ has equation $2xx_0-3yy_0=5$. If this tangent is parallel to $8x=9y$, then $\frac{2x_0}{3y_0}=\frac{8}{9} \Rightarrow y_0=\frac{3}{4}x_0$.

Since the point $P(x_0,y_0)$ lies on the hyperbola, then $2x_0^2-3y_0^2=5$. Therefore $2x_0^2-3\cdot\frac{9}{16}x_0^2=5 \Rightarrow x_0^2=16$. Hence the tangents to the hyperbola $2x^2-3y^2=5$ are

$8x-9y=5$, with point of contact $P(4,3)$ and

$8x-9y=-5$, with point of contact $P(-4,-3)$.

9 Solution

The tangent to the hyperbola $x^2-y^2=7$ at the point $P(x_0,y_0)$ has equation $xx_0-yy_0=7$. If this tangent is parallel to $3y=4x$, then $\frac{x_0}{y_0}=\frac{4}{3} \Rightarrow y_0=\frac{3}{4}x_0$. Since the point $P(x_0,y_0)$ lies on the hyperbola, then $x_0^2-y_0^2=7$. Therefore $x_0^2-\frac{9}{16}x_0^2=7 \Rightarrow x_0^2=16$. Hence the tangents to the hyperbola $x^2-y^2=7$ are

$4x-3y=7$, with point of contact $P(4,3)$ and

$4x-3y=-7$, with point of contact $P(-4,-3)$.

10 Solution

The tangent to the ellipse $8x^2 + 3y^2 = 35$ at the point $P(x_0, y_0)$ has equation $8xx_0 + 3yy_0 = 35$. The point $\left(\frac{5}{4}, 5\right)$ lies on this tangent. So $10x_0 + 15y_0 = 35 \Rightarrow y_0 = \frac{7}{3} - \frac{2}{3}x_0$. Since the point $P(x_0, y_0)$ lies on the ellipse, then $8x_0^2 + 3y_0^2 = 35$.

Therefore $8x_0^2 + 3 \cdot \left(\frac{7}{3} - \frac{2}{3}x_0\right)^2 = 35 \Rightarrow 28x_0^2 - 28x_0 - 56 = 0 \Rightarrow (x_0 - 2)(x_0 + 1) = 0$.

Hence the tangents to the ellipse $8x^2 + 3y^2 = 35$ from the point $\left(\frac{5}{4}, 5\right)$ are

$16x + 3y = 35$, with point of contact $P(2,1)$ and

$-8x + 9y = 35$, with point of contact $P(-1,3)$.

11 Solution

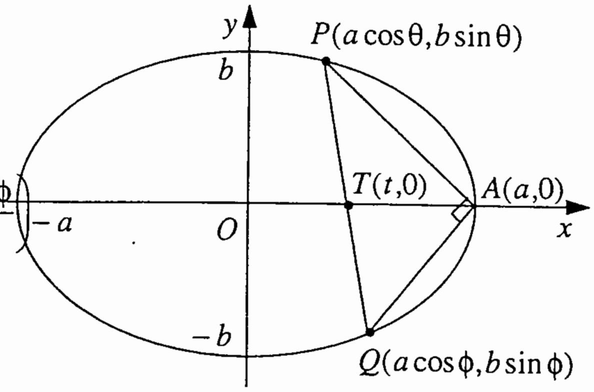
The tangent to the hyperbola $x^2 - 9y^2 = 9$ at the point $P(x_0, y_0)$ has equation $xx_0 - 9yy_0 = 9$. The point $(3,2)$ lies on this tangent. So $3x_0 - 18y_0 = 9 \Rightarrow x_0 = 3 + 6y_0$. Since the point $P(x_0, y_0)$ lies on the hyperbola, then $x_0^2 - 9y_0^2 = 9$. Therefore $(3 + 6y_0)^2 - 9y_0^2 = 9 \Rightarrow 3y_0^2 + 4y_0 = 0 \Rightarrow y_0(3y_0 + 4) = 0$.

Hence the tangents to the hyperbola $x^2 - 9y^2 = 9$ from the point $(3,2)$ are

$x = 3$, with point of contact $P(3,0)$ and

$-5x + 12y = 9$, with point of contact $P\left(-5, -\frac{4}{3}\right)$.

12 Solution

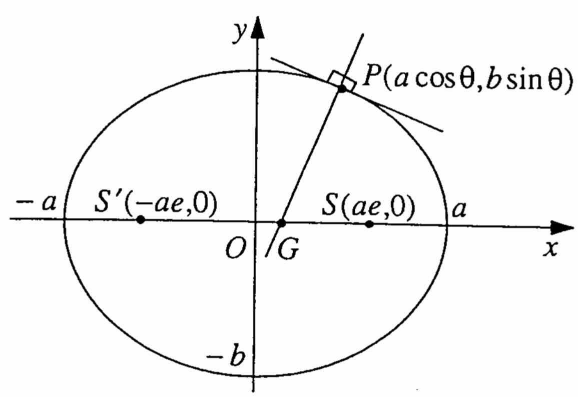


The chord PQ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has equation $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$, where P, Q have parameters θ, ϕ. The chord PQ cuts the x-axis at point $T(t,0)$. So $t=a\cos\left(\frac{\theta-\phi}{2}\right)\sec\left(\frac{\theta+\phi}{2}\right)=a\left(1+\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)\left(1-\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)^{-1}$.

The gradient of AP is $\frac{b\sin\theta}{a(\cos\theta-1)}=-\frac{b}{a}\cot\frac{\theta}{2}$ and the gradient of AQ is $\frac{b\sin\theta}{a(\cos\phi-1)}=-\frac{b}{a}\cot\frac{\phi}{2}$. If the chord PQ subtends a right angle at the point A, then gradient $AP\times$ gradient $AQ=-1$. Therefore $\frac{b^2}{a^2}\cot\frac{\theta}{2}\cot\frac{\phi}{2}=-1\Rightarrow\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$.

Hence $t=a\left(1-\frac{b^2}{a^2}\right)\left(1+\frac{b^2}{a^2}\right)^{-1}=a\frac{a^2-b^2}{a^2+b^2}$. But for the ellipse $b^2=a^2(1-e^2)$. Thus $t=\frac{ae^2}{2+e^2}$. So PQ passes through a fixed point $T\left(\frac{ae^2}{2+e^2},0\right)$ on the x-axis.

13 Solution



The normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ has equation $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$. The point G has coordinates $\left(\frac{a^2-b^2}{a}\cos\theta,0\right)$. Therefore

$$PG^2 = \left(a - \frac{a^2 - b^2}{a}\right)^2 \cos^2\theta + b^2\sin^2\theta = \frac{b^2}{a^2}\left(b^2\cos^2\theta + a^2\sin^2\theta\right).$$

But for the ellipse $b^2 = a^2(1-e^2)$. Hence $PG^2 = a^2(1-e^2)(1-e^2\cos^2\theta)$.

From the other side

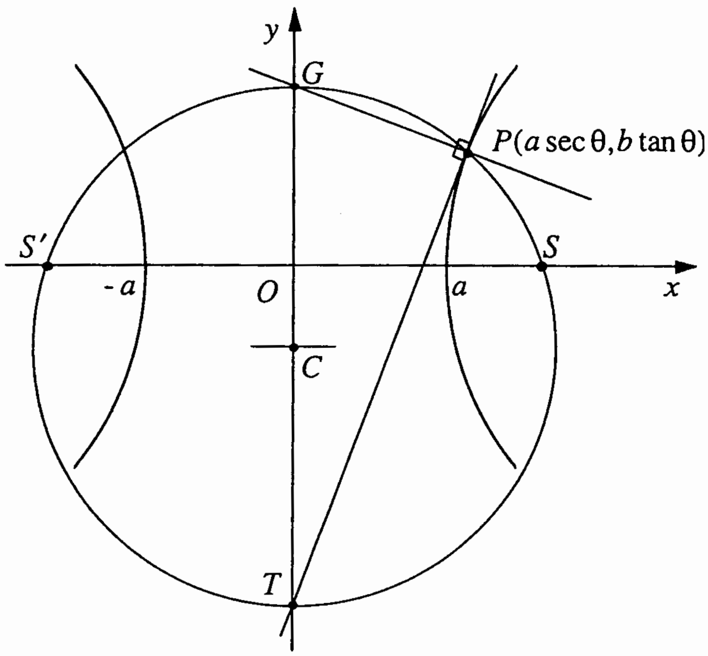
$$PS^2 = a^2(e-\cos\theta)^2 + b^2\sin^2\theta = a^2(1-2e\cos\theta + e^2\cos^2\theta) = a^2(1-e\cos\theta)^2,$$

$$PS'^2 = a^2(e+\cos\theta)^2 + b^2\sin^2\theta = a^2(1+2e\cos\theta + e^2\cos^2\theta) = a^2(1+e\cos\theta)^2.$$

Thus $PG^2 = (1-e^2)\cdot a(1-e\cos\theta)\cdot a(1+e\cos\theta) = (1-e^2)PS\cdot PS'$.

14 Solution

The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. The point T has coordinates $(0, -b\cot\theta)$. The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$. The point G has coordinates $\left(0, \frac{a^2+b^2}{b}\tan\theta\right)$. So gradient $SG \times$ gradient $ST = \frac{a^2+b^2}{-bae}\tan\theta \cdot \frac{-b\cot\theta}{-ae}$. Since for the hyperbola $b^2 = a^2(e^2-1)$, then gradient $SG \times$ gradient $ST = -\frac{a^2+b^2}{a^2e^2} = -1$. Thus $SG \perp ST$ and consequently GT subtends a right angle at focus S. Similarly gradient $S'G \times$ gradient $S'T = \frac{a^2+b^2}{bae}\tan\theta \cdot \frac{-b\cot\theta}{ae} = -\frac{a^2+b^2}{a^2e^2} = -1$. Thus $S'G \perp S'T$ and consequently GT subtends a right angle at focus S'. Therefore S, G, S', T are concyclic with GT the diameter of the circle through the points.



15 Solution

The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ has equation $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$. So the point N has coordinates $\left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$. Since the asymptotes have equations $y=\pm\frac{b}{a}x$, then the point Q has coordinates $(a\sec\theta, \pm b\sec\theta)$. Thus the gradient of QN is $\mp b\sec\theta\cdot\left[\left(\frac{a^2+b^2}{a}-a\right)\sec\theta\right]^{-1}=\mp\frac{a}{b}$. Therefore QN is perpendicular to the asymptote.

16 Solution

Let φ denotes the smallest angle from positive x-axis to the asymptote $y=\frac{b}{a}x$.

Then $\alpha=2\varphi$ when $\varphi\le\frac{\pi}{4}$, or $\alpha=\pi-2\varphi$ when $\varphi>\frac{\pi}{4}$. Therefore $\tan\alpha=|\tan 2\varphi|$.

Since $\tan\varphi=\frac{b}{a}$, then $\tan\alpha=\left|\frac{2\tan\varphi}{1-\tan^2\varphi}\right|=\left|\frac{2b}{a}\cdot\left(1-\frac{b^2}{a^2}\right)^{-1}\right|=\frac{2ab}{|a^2-b^2|}$.

17 Solution

The normal to the hyperbola $xy=c^2$ at the point $P\left(ct,\frac{c}{t}\right)$ has equation $tx-\frac{y}{t}=c\left(t^2-\frac{1}{t^2}\right)$. Let the point Q, R have coordinates (x_1,y_1) and (x_2,y_2) respectively. Since Q, R lie on the hyperbola $x^2-y^2=a^2$, then

$$(x_1^2-x_2^2)-(y_1^2-y_2^2)=0\Rightarrow(x_1-x_2)(x_1+x_2)=(y_1-y_2)(y_1+y_2).$$

(1)

The points Q, R lie on the normal to the hyperbola. Therefore

$$t(x_1-x_2)-\frac{y_1-y_2}{t}=0,$$

(2)

$$t(x_1 + x_2) - \frac{y_1 + y_2}{t} = 2c\left(t^2 - \frac{1}{t^2}\right).$$

(3)

Substituting (2) into (1), we obtain

$$x_1 + x_2 = t^2(y_1 + y_2).$$

(4)

Then $$(3),(4) \Rightarrow t^2(y_1 + y_2) - \frac{1}{t^2}(y_1 + y_2) = \frac{2c}{t}\left(t^2 - \frac{1}{t^2}\right) \Rightarrow y_1 + y_2 = \frac{2c}{t}.$$

(5)

Using (5) we get from (4)

$$x_1 + x_2 = 2ct.$$

(6)

Thus, according to (5) and (6), the midpoint of QR has coordinates $\left(ct, \frac{c}{t}\right)$. Hence the point $P\left(ct, \frac{c}{t}\right)$ is the midpoint of QR.

18 Solution

The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ has equation $tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right)$. The point $Q\left(cq, \frac{c}{q}\right)$ lies on the normal. Hence $tcq - \frac{c}{tq} = c\left(t^2 - \frac{1}{t^2}\right) \Rightarrow (tq - t^2)\left(1 + \frac{1}{t^3 q}\right) = 0$. Since $Q \neq P$, then $q \neq t$. Therefore $q = -\frac{1}{t^3}$ and Q has coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$. The point $R\left(cr, \frac{c}{r}\right)$ lies on the circle on PQ as diameter. Hence gradient $RP \times$ gradient $RQ = -1$. But gradient of RP is $c\left(\frac{1}{r} - \frac{1}{t}\right) \cdot \frac{1}{c(r-t)} = -\frac{1}{rt}$ and gradient of RQ is $c\left(\frac{1}{r} - \frac{1}{q}\right) \cdot \frac{1}{c(r-q)} = -\frac{1}{rq}$. Thus $\frac{1}{r^2 tq} = -1 \Rightarrow r^2 = -\frac{1}{tq}$. Since $q = -\frac{1}{t^3}$, then $r^2 = t^2$. Therefore $r = -t$, because

$R \neq P$. So the point R has coordinates $\left(-ct, -\dfrac{c}{t}\right)$.

19 Solution

If $M(x,y)$ is the midpoint of AP, then $x = \dfrac{a}{2}(\sec\theta + 1)$ and $y = \dfrac{a}{2}\tan\theta$. Therefore $(2x-a)^2 - (2y)^2 = a^2(\sec^2\theta - \tan^2\theta) = a^2$. Hence the locus of M is hyperbola $(2x-a)^2 - (2y)^2 = a^2$.

20 Solution

The normal to the hyperbola $xy = c^2$ at the point $P\left(ct, \dfrac{c}{t}\right)$ has equation $tx - \dfrac{y}{t} = c\left(t^2 - \dfrac{1}{t^2}\right)$. The point $Q\left(cq, \dfrac{c}{q}\right)$ lies on the normal. Hence $tcq - \dfrac{c}{tq} = c\left(t^2 - \dfrac{1}{t^2}\right) \Rightarrow (tq - t^2)\left(1 + \dfrac{1}{t^3 q}\right) = 0$. Since $Q \neq P$, then $q \neq t$. Therefore $q = -\dfrac{1}{t^3}$ and Q has coordinates $\left(-\dfrac{c}{t^3}, -ct^3\right)$. If $M(x,y)$ is the midpoint of PQ, then

$$x = \frac{c}{2}(t+q) = \frac{c}{2t}\left(t^2 - \frac{1}{t^2}\right) \tag{7}$$

and

$$y = \frac{c}{2}\left(\frac{1}{t} + \frac{1}{q}\right) = \frac{ct}{2}\left(\frac{1}{t^2} - t^2\right). \tag{8}$$

We obtain from $(7), (8)$ that $\dfrac{2tx}{c} = -\dfrac{2y}{ct} \Rightarrow t^2 = -\dfrac{y}{x}$. Substituting this formula for t^2 into (7), we get

$$x = \frac{c}{2\sqrt{-y/x}}\left(-\frac{y}{x} + \frac{x}{y}\right) \Rightarrow x^2 = \frac{-c^2 x}{4y} \cdot \frac{(x^2 - y^2)}{x^2 y^2} \Rightarrow 4x^3y^3 + c^2(x^2 - y^2)^2 = 0.$$

Therefore the locus of M has equation $4x^3y^3 + c^2(x^2 - y^2)^2 = 0$.