7SD Solutions Series

Worked Solutions to Popular Mathematics Texts

Suggested Worked Solutions to

"4 Unit Mathematics"

(Text book for the NSW HSC by D. Arnold and G. Arnold)

Chapter 5
Integration

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Solutions are to “4 Unit Mathematics”

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Exercise 5.1

1 *Solution*

Using the pattern $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$ with $f(x) = 1 + x^2$, we have

$$\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{2xdx}{1+x^2} = \frac{1}{2}\ln|1+x^2| + c = \frac{1}{2}\ln(1+x^2) + c,$$

since $1 + x^2 > 0$.

2 *Solution*

Using the pattern $\int \{f(x)\}^n f'(x)dx = \frac{1}{n+1}\{f(x)\}^{n+1} + c,\ n = -2$ with $f(x) = 1 + x^2$, we have

$$\int \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int \frac{2x}{(1+x^2)^2}dx = -\frac{1}{2(1+x^2)} + c.$$

3 *Solution*

The given integral follows the pattern $\int e^{f(x)}f'(x)dx = e^{f(x)} + c$ with $f(x) = \sin x$, and we have

$$\int e^{\sin x}\cos x dx = e^{\sin x} + c.$$

4 *Solution*

The given integral follows the pattern $\int \sin\{f(x)\}f'(x)dx = -\cos\{f(x)\} + c$ with $f(x) = e^x$,

and we get

$$\int e^x \sin(e^x)dx = -\cos(e^x) + c.$$

5 *Solution*

The given integral follows the pattern $\int \{f(x)\}^n f'(x)dx = \frac{1}{n+1}\{f(x)\}^{n+1} + c$ with

$f(x) = 1 + x^2$, $n = 1/2$, and we have

$$\int x\sqrt{(1+x^2)}dx = \frac{1}{2}\int (1+x^2)^{1/2}2xdx = \frac{1}{3}(1+x^2)^{3/2} + c.$$

6 *Solution*

The given integral follows the pattern $\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}(\frac{x}{a}) + c$ with $a = 2$, and we get

$$\int \frac{1}{\sqrt{4 - x^2}}dx = \sin^{-1}(\frac{x}{2}) + c.$$

7 *Solution*

The given integral follows the pattern $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ with $a = \frac{1}{2}$, and we get

$$\int \frac{dx}{1+4x^2} = \frac{1}{4}\int \frac{dx}{\frac{1}{4}+x^2} = \frac{1}{2}\tan^{-1}2x + c.$$

8 *Solution*

The given integral follows the pattern $\int \{f(x)\}^3 f'(x)dx = \frac{1}{4}\{f(x)\}^4 + c$ with $f(x) = \tan x$, and we get

$$\int \tan^3 x \sec^2 x dx = \frac{1}{4}\{\tan x\}^4 + c.$$

9 *Solution*

Using $\int \sec^2\{f(x)\}f'(x)dx = \tan\{f(x)\} + c$ with $f(x) = x^2$, we get

$$\int x\sec^2(x^2)dx = \frac{1}{2}\int \sec^2(x^2)2xdx = \frac{1}{2}\tan(x^2) + c.$$

10 *Solution*

Using $\int e^{f(x)}f'(x)dx = e^{f(x)} + c$ with $f(x) = \sqrt{x}$, we have $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^{\sqrt{x}}(\sqrt{x})'dx = 2e^{\sqrt{x}} + c$.

11 *Solution*

Using the pattern $\int \{f(x)\}^n f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1} + c$ with $f(x) = \cos x$ and $n = -4$, we get

$$\int \sec^3 x \tan x dx = \int \frac{\sin x dx}{\cos^4 x} = -\int (\cos x)^{-4}(-\sin x)dx = \frac{(\cos x)^{-3}}{3} + c = \frac{\sec^3 x}{3} + c.$$

12 *Solution*

Using $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$ with $f(x) = \sin^2 x + 2$, we get

$$\int \frac{\sin 2x}{2+\sin^2 x}dx = \int \frac{2\sin x\cos x}{2+\sin^2 x}dx = \ln\left|\sin^2 x + 2\right| + c = \ln(\sin^2 x + 2) + c,$$ since $\sin^2 x + 2 > 0$.

13 *Solution*

Using the pattern $\int \{f(x)\}^{-1} f'(x)dx = \ln|f(x)| + c$ with $f(x) = \sin x$, we get

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \cot x dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} (\sin x)^{-1} \cos x dx = \left[\ln|\sin x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{6}} = \ln(\sin\frac{\pi}{6}) - \ln(\sin\frac{\pi}{4})$$

$$= \ln(\frac{1}{2}) - \ln(\frac{1}{\sqrt{2}}) = \ln(\frac{\sqrt{2}}{2}) = \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2.$$

14 *Solution*

Using the pattern $\int \cos\{f(x)\} f'(x)dx = \sin\{f(x)\} + c$ with $f(x) = \ln x$, we get

$$\int_{1}^{e} \cos(\ln x)\frac{1}{x} dx = \sin(\ln x)\Big|_{1}^{e} = \sin(\ln e) - \sin(\ln 1) = \sin(1) - \sin(0) = \sin 1.$$

15 *Solution*

Using the pattern $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$ with $a = 2$, we obtain

$$\int_{0}^{2} \frac{1}{4 + x^2} dx = \frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_{0}^{2} = \frac{1}{2}\tan^{-1}\frac{2}{2} - \frac{1}{2}\tan^{-1} 0 = \frac{1}{2}\tan^{-1}(1) = \frac{\pi}{8}.$$

16 *Solution*

Using the pattern $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \ln\left|x + \sqrt{(x^2 - a^2)}\right| + c$ with $a = 1$, we obtain

$$\int_{\sqrt{2}}^{3} \frac{1}{\sqrt{(x^2 - 1)}} dx = \ln\left|x + \sqrt{x^2 - 1}\right|\Big|_{\sqrt{2}}^{3} = \ln(3 + \sqrt{8}) - \ln(\sqrt{2} + \sqrt{1}) = \ln(\frac{3 + 2\sqrt{2}}{\sqrt{2} + 1}) = \ln(1 + \sqrt{2}).$$

17 *Solution*

Using the pattern $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\{x + \sqrt{x^2 + a^2}\} + c$ with $a = 2$, we obtain

$$\int_{0}^{2} \frac{1}{\sqrt{x^2 + 4}} dx = \left[\ln(x + \sqrt{x^2 + 4})\right]_{0}^{2} = \ln(2 + \sqrt{8}) - \ln(2) = \ln(1 + \sqrt{2}).$$

18 *Solution*

Using the pattern $\int \frac{1}{\sqrt{(a^2-x^2)}}dx = \sin^{-1}\frac{x}{a}$ with $a = 1/2$, we obtain

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1-4x^2)}}dx = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{\frac{1}{4}-x^2}}dx = \frac{1}{2}\sin^{-1}(2x)\Big|_0^{\frac{1}{2}} = \frac{1}{2}\sin^{-1}(1) = \frac{\pi}{4}.$$

19 *Solution*

Using the pattern $\int \{f(x)\}^{-2} f'(x)dx = -\{f(x)\}^{-1} + c$ with $f(x) = \cos 2x$, we get

$$\int_0^{\frac{\pi}{6}} \tan 2x \sec 2x dx = \int_0^{\frac{\pi}{6}} \frac{\sin 2x}{\cos^2 2x}dx = -\frac{1}{2}\int \frac{(\cos 2x)'}{\cos^2 2x}dx = \frac{1}{2}\left[\frac{1}{\cos 2x}\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\left(\frac{1}{\cos(\frac{\pi}{3})} - \frac{1}{\cos 0}\right) = \frac{1}{2}.$$

20 *Solution*

Using $\int \{f(x)\}^{-1} f'(x)dx = \ln|f(x)| + c$ with $f(x) = 1 + e^x$, we have

$$\int_0^{\ln 3} \frac{e^x}{1+e^x}dx = \left[\ln(1+e^x)\right]_0^{\ln 3} = \ln(1+e^{\ln 3}) - \ln(1+e^0) = \ln 4 - \ln 2 = \ln 2,$$

since $1 + e^x > 0$.

Exercise 5.2

1 *Solution*

It is clear that $x^2+2x+2=(x+1)^2+1$. Make the substitution $x+1=u, dx=du$. Then we get

$$\int\frac{1}{x^2+2x+2}dx=\int\frac{1}{(x+1)^2+1}dx=\int\frac{du}{u^2+1}=\tan^{-1}u+c=\tan^{-1}(x+1)+c.$$

2 *Solution*

It is clear that $2x-x^2=1-(x-1)^2$. Make the substitution $x-1=u, dx=du$. Then we get

$$\int\frac{1}{\sqrt{2x-x^2}}dx=\int\frac{1}{\sqrt{1-(x-1)^2}}dx=\int\frac{du}{\sqrt{1-u^2}}=\sin^{-1}u+c=\sin^{-1}(x-1)+c.$$

3 *Solution*

It is easily seen that

$\int\frac{x-1}{x^2+1}dx=\int\frac{x}{x^2+1}dx-\int\frac{dx}{x^2+1}=\frac{1}{2}\int\frac{2x}{x^2+1}dx-\int\frac{dx}{x^2+1}=\frac{1}{2}\ln(x^2+1)-\tan^{-1}x+c$, since

$(x^2+1)'=2x$.

4 *Solution*

It is easily seen that

$$\frac{x(2x+1)}{x+1}=\frac{(x+1-1)(2x+2-1)}{x+1}=\frac{2(x+1)^2-3(x+1)+1}{x+1}=2(x+1)-3+\frac{1}{x+1}=2x-1+\frac{1}{x+1}.$$

Then we have

$$\int\frac{x(2x+1)}{x+1}dx=\int(2x-1+\frac{1}{x+1})dx=\int 2xdx-\int dx+\int\frac{1}{x+1}dx=x^2-x+\ln|x+1|+c.$$

5 *Solution*

Let $\frac{x+1}{(2x+1)x}\equiv\frac{a}{x}+\frac{b}{2x+1}$, a,b constants.

Then $x+1\equiv a(2x+1)+bx$.

Put $x=0$: $a=1$.

Put $x=1$: $b=-1$.

Then we have

$$\int\frac{x+1}{x(2x+1)}dx=\int\left(\frac{1}{x}-\frac{1}{2x+1}\right)dx=\int\frac{1}{x}dx-\int\frac{1}{2x+1}dx=\ln|x|-\frac{1}{2}\ln|2x+1|+c=\ln\left(\left|\frac{x}{\sqrt{2x+1}}\right|\right)+c$$

6 *Solution*

By division $\dfrac{x^2}{(x+1)(x+2)}=1-\dfrac{3x+2}{(x+1)(x+2)}$.

Let $\dfrac{3x+2}{(x+1)(x+2)}\equiv\dfrac{a}{x+1}+\dfrac{b}{x+2}$.

Then we get $3x+2\equiv a(x+2)+b(x+1)$.

Put $x=-1$: $a=-1$.

Put $x=-2$: $b=4$.

Hence

$$\int\frac{x^2}{(x+1)(x+2)}dx=\int 1dx-\int\frac{3x+2}{(x+1)(x+2)}dx=x-\int\left(-\frac{1}{x+1}+\frac{4}{x+2}\right)dx$$

$$=x+\int\frac{1}{x+1}dx-4\int\frac{1}{x+2}dx=x+\ln|x+1|-4\ln|x+2|+c=x+\ln\left(\frac{|x+1|}{(x+2)^4}\right)+c.$$

7 *Solution*

It is easily seen that

$2x+3=2x+2+1=(x^2+2x+5)'+1$.

Hence

$$\int\frac{2x+3}{x^2+2x+5}dx=\int\frac{(x^2+2x+5)'}{x^2+2x+5}dx+\int\frac{1}{x^2+2x+5}dx=\ln\left|x^2+2x+5\right|+\int\frac{1}{(x+1)^2+2^2}dx$$

$$=\ln(x^2+2x+5)+\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c,$$

since $x^2+2x+5>0,\ x\in R$.

8 *Solution*

Let $\dfrac{6x-10}{(x+1)(x+3)}\equiv\dfrac{a}{x+1}+\dfrac{b}{x-3}$.

Then ,we have $6x-10 \equiv a(x-3)+b(x+1)$.

Put $x=-1: a=4$.

Put $x=3: b=2$.

Hence $\int \frac{6x-10}{(x+1)(x-3)}dx = \int\left(\frac{4}{x+1}+\frac{2}{x-3}\right)dx = 4\int \frac{1}{x+1}dx + 2\int \frac{1}{x-3}dx$

$= 4\ln|x+1| + 2\ln|x-3| + c = \ln\left((x+1)^4(x-3)^2\right)+c$.

9 *Solution*

Make the substitution $x-1=u, dx=du$.

Then $\int \frac{4}{x^2-2x-1}dx = 4\int \frac{1}{(x^2-2x+1)-2}dx = 4\int \frac{1}{(x-1)^2-2}dx = 4\int \frac{1}{u^2-2}dx$

$= \frac{4}{2\sqrt{2}}\int\left(\frac{1}{u-\sqrt{2}}-\frac{1}{u+\sqrt{2}}\right)du = \sqrt{2}\left(\int \frac{1}{u-\sqrt{2}}du - \int \frac{1}{u+\sqrt{2}}du\right) = \sqrt{2}(\ln|u-\sqrt{2}| - \ln|u+\sqrt{2}|)+c$

$= \sqrt{2}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + c = \sqrt{2}\ln\left|\frac{x-1-\sqrt{2}}{x-1+\sqrt{2}}\right| + c$.

10 *Solution*

Let $\frac{4x-x^2}{(x+1)(x^2+4)} \equiv \frac{a}{x+1}+\frac{bx+c}{x^2+4}$.

Then $4x-x^2 \equiv a(x^2+4)+(bx+c)(x+1)$.

Put $x=-1$: $-5=5a \Rightarrow a=-1$.

Equate coefficients of $x^2: -1=a+b \Rightarrow b=0$.

Equate coefficients of x^1: $4=b+c \Rightarrow c=4$.

Hence

$\int \frac{4x-x^2}{(x+1)(x^2+4)}dx = \int\left(\frac{-1}{x+1}+\frac{4}{x^2+4}\right)dx = \int \frac{-1}{x+1}dx + \int \frac{4}{x^2+4}dx = -\ln|x+1| + 2\tan^{-1}(\frac{x}{2})+c$.

11 *Solution*

Let $\frac{10}{(x-1)(x^2+9)} \equiv \frac{a}{x-1}+\frac{bx+c}{x^2+9}$.

Then $10 \equiv a(x^2+9)+(bx+c)(x-1)$.

Equate coefficients of x^2: $0 = a + b$.

Equate coefficients of x^1: $0 = c - b$.

Equate constant terms: $10 = 9a - c$.

Thus we get $a = 1$, $b = -1$, $c = -1$.

Hence

$$\int \frac{10}{(x-1)(x^2+9)}dx = \int\left(\frac{1}{x-1} - \frac{x+1}{x^2+9}\right)dx = \int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{2x}{x^2+9}dx - \int \frac{1}{x^2+9}dx$$

$$= \ln|x-1| - \frac{1}{2}\ln\left|x^2+9\right| - \frac{1}{3}\tan^{-1}\frac{x}{3} + c = \ln\left|\frac{x-1}{\sqrt{x^2+9}}\right| - \frac{1}{3}\tan^{-1}\frac{x}{3} + c,$$

since $x^2 + 9 > 0,\ x \in R$.

12 *Solution*

Let $\dfrac{3}{(x^2+1)(x^2+4)} \equiv \dfrac{ax+b}{x^2+1} + \dfrac{cx+d}{x^2+4}$.

Then $3 \equiv (ax+b)(x^2+4) + (cx+d)(x^2+1)$.

Equate coefficients of x^3: $0 = a + c$.

Equate coefficients of x^2: $0 = b + d$.

Equate coefficients of x^1: $0 = 4a + c$.

Equate constant terms: $3 = 4b + d$.

Thus we obtain $a = c = 0$, $b = 1$, $d = -1$.

Hence

$$\int \frac{3}{(x^2+1)(x^2+4)}dx = \int\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right)dx = \int \frac{1}{x^2+1}dx - \int \frac{1}{x^2+4}dx = \tan^{-1}x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right.$$

13 *Solution*

Using the substitution $x - 1 = u, dx = du$, $x = -1 \Rightarrow u = -2$, $x = 3 \Rightarrow u = 2$ we get

$$\int_{-1}^{3} \frac{1}{x^2-2x+5}dx = \int_{-1}^{3} \frac{1}{(x-1)^2+4}dx = \int_{-2}^{2} \frac{1}{u^2+4}du = \frac{1}{2}\tan\frac{u}{2}\Big|_{-2}^{2} = \frac{1}{2}(\tan 1 - \tan(-1))$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{4}.$$

14 *Solution*

Using the substitution $x+1=u, dx=du, x=0 \Rightarrow u=1, x=-1 \Rightarrow u=0$, we get

$$\int_{-1}^{0} \frac{1}{\sqrt{3-2x-x^2}}dx = \int_{-1}^{0} \frac{1}{\sqrt{4-(1+2x+x^2)}}dx = \int_{-1}^{0} \frac{1}{\sqrt{4-(x+1)^2}}dx = \int_{0}^{1} \frac{1}{\sqrt{4-u^2}}du$$

$$= \left[\sin^{-1}\frac{u}{2}\right]_0^1 = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6}.$$

15 *Solution*

$$\int_{0}^{2} \frac{x+1}{x^2+4}dx = \int_{0}^{2} \frac{x}{x^2+4}dx + \int_{0}^{2} \frac{1}{x^2+4}dx = \frac{1}{2}\left[\ln(x^2+4)\right]_0^2 + \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{1}{2}(\ln 8 - \ln 4) + \frac{1}{2}(\tan^{-1}(1) - \tan^{-1}(0)) = \frac{1}{2}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{1}{2}\ln 2 + \frac{\pi}{8}.$$

16 *Solution*

By division $\dfrac{7+x-2x^2}{2-x} = 2x+3+\dfrac{1}{2-x}$.

Then $\displaystyle\int_{0}^{1} \frac{7+x-2x^2}{2-x}dx = \int_{0}^{1} (2x+3)dx + \int_{0}^{1} \frac{1}{2-x}dx = \left[x^2+3x\right]_0^1 - \left[\ln|2-x|\right]_0^1 = 4 - (\ln 1 - \ln 2)$

$= 4 + \ln 2$.

17 *Solution*

Make the substitution $x-1=u, dx=du, x=1 \Rightarrow u=0, x=2 \Rightarrow u=1, x=u+1$.

Hence

$$\int_{1}^{2} \frac{2x-3}{x^2-2x+2}dx = \int_{1}^{2} \frac{2x-3}{(x-1)^2+1}dx = \int_{0}^{1} \frac{2u-1}{u^2+1}du = \int_{0}^{1} \frac{2u}{u^2+1}du - \int_{0}^{1} \frac{du}{u^2+1} = \left[\ln(u^2+1)\right]_0^1 - \left[\tan^{-1}u\right]_0^1$$

$$\ln 2 - \ln 1 - (\tan^{-1}1 - \tan^{-1}0) = \ln 2 - \frac{\pi}{4}.$$

18 *Solution*

By division $\dfrac{x^2+4x+5}{(x+1)(x+3)} \equiv 1 + \dfrac{2}{(x+1)(x+3)}$.

Let $\dfrac{2}{(x+1)(x+3)} \equiv \dfrac{a}{(x+1)} + \dfrac{b}{(x+3)}$, a, b constants.

Then $2 \equiv a(x+3)+b(x+1)$.

Put $x=-1$: $\quad 2=2a \Rightarrow a=1$.

Put $x=-3$: $\quad 2=-2b \Rightarrow b=-1$.

Thus we get

$$\int_0^3 \frac{x^2+4x+5}{(x+1)(x+3)}dx=\int_0^3 1dx+\int_0^1 \frac{2}{(x+1)(x+3)}dx=\left[x\right]_0^3+\int_0^3 \frac{1}{x+1}dx-\int_0^3 \frac{1}{x+3}dx$$

$$=3+\left[\ln|x+1|\right]_0^3-\left[\ln|x+3|\right]_0^3=3+\ln 4-\ln 1-(\ln 6-\ln 3)=3+\ln 2.$$

19 *Solution*

Let $\dfrac{1+4x}{(x^2+1)(4-x)} \equiv \dfrac{a}{4-x}+\dfrac{bx+c}{x^2+1}$, a, b, c constants.

Then $1+4x \equiv a(x^2+1)+(bx+c)(4-x)$.

Equate coefficients of x^2: $\quad 0=a-b$.

Equate coefficients of x^1: $\quad 4=4b-c$.

Equate constant terms x^0: $\quad 1=a+4c$.

Thus we get $a=1, b=1, c=0$.

Hence

$$\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}dx=\int_0^2 \frac{1}{4-x}dx+\int_0^2 \frac{x}{x^2+1}dx=-\left[\ln|4-x|\right]_0^2+\frac{1}{2}\left[\ln(x^2+1)\right]_0^2=-(\ln 2-\ln 4)$$

$$+\frac{1}{2}(\ln 5-\ln 1)=\ln 2+\frac{1}{2}\ln 5=\frac{1}{2}\ln 20.$$

20 *Solution*

Let $\dfrac{8}{(x^2+1)(x^2+9)} \equiv \dfrac{ax+b}{x^2+1}+\dfrac{cx+d}{x^2+9}$.

Then $8 \equiv (ax+b)(x^2+9)+(cx+d)(x^2+1)$.

Equate coefficients of x^3: $\quad 0=a+c$.

Equate coefficients of x^2: $\quad 0=b+d$.

Equate coefficients of x^1: $\quad 0=9a+c$.

Equate constant terms: $\quad 8=9b+d$.

Thus we get $a=0, c=0, b=1, d=-1$.

Hence $\int_0^{\sqrt{3}} \frac{8}{(x^2+1)(x^2+9)}dx = \int_0^{\sqrt{3}} \frac{1}{x^2+1}dx - \int_0^{\sqrt{3}} \frac{1}{x^2+9}dx = \left[\tan^{-1} x\right]_0^{\sqrt{3}} - \frac{1}{3}\left[\tan^{-1}(\frac{x}{3})\right]_0^{\sqrt{3}}$

$=\frac{\pi}{3}-\frac{\pi}{18}=\frac{5\pi}{18}$.

Exercise 5.3

1 *Solution*

Let $u=\sqrt{x}$. Then $x=u^2$, $dx=2udu$, and we get

$$\int\frac{1}{\sqrt{x}\sqrt{1-x}}dx=\int\frac{1}{u\sqrt{1-u^2}}2udu=2\int\frac{1}{\sqrt{1-u^2}}du=2\sin^{-1}u+c=2\sin^{-1}(\sqrt{x})+c.$$

2 *Solution*

Let $u=x^2$. Then $x=\sqrt{u}$, $dx=\frac{1}{2\sqrt{u}}du$.

Hence

$$\int\frac{x}{x^4-1}dx=\int\frac{\sqrt{u}}{u^2-1}\frac{1}{2\sqrt{u}}du=\frac{1}{2}\int\frac{1}{u^2-1}du=\frac{1}{4}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du=\frac{1}{4}\int\frac{1}{u-1}du-\frac{1}{4}\int\frac{1}{u+1}du$$

$$=\frac{1}{4}\ln|u-1|-\frac{1}{4}\ln|u+1|+c=\frac{1}{4}\ln\left|\frac{u-1}{u+1}\right|+c=\frac{1}{4}\ln\left|\frac{x^2-1}{x^2+1}\right|+c.$$

3 *Solution*

Let $u^2=x+1$. Then $x=u^2-1$, $\sqrt{x+1}=u$, $dx=2udu$.

Hence

$$\int x\sqrt{x+1}dx=\int(u^2-1)u2udu=2\int(u^4-u^2)du=\frac{2}{5}u^5-\frac{2}{3}u^3+c=\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+c.$$

4 *Solution*

Let $u^2=x-1$. Then $x=u^2+1$, $\sqrt{x-1}=u$, $dx=2udu$.

Hence $\int x^2\sqrt{x-1}dx=\int(u^2+1)^2u2udu=2\int(u^6+2u^4+u^2)du=\frac{2}{7}u^7+\frac{4}{5}u^5+\frac{2}{3}u^3+c$

$$=\frac{2}{7}(x-1)^{7/2}+\frac{4}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+c.$$

5 *Solution*

Let $u=e^x$, $u>0$. Then $x=\ln u$, $dx=\frac{1}{u}du$.

Hence $\int\frac{1}{e^x+1}dx=\int\frac{1}{u+1}\frac{1}{u}du=\int\left(\frac{1}{u}-\frac{1}{u+1}\right)du=\int\frac{1}{u}du-\int\frac{1}{u+1}du=\ln u-\ln(u+1)+c$

$= x - \ln(e^x + 1) + c$.

6 *Solution*

Let $u = e^x, u > 0$. Then $x = \ln u, dx = \frac{1}{u}du$.

Hence

$$\int \frac{e^x + e^{2x}}{1 + e^{2x}}dx = \int \frac{u + u^2}{1 + u^2}\frac{1}{u}du = \int \frac{1 + u}{1 + u^2}du = \int \frac{1}{1 + u^2}du + \int \frac{u}{1 + u^2}du = \tan^{-1} u + \frac{1}{2}\ln(u^2 + 1) + c$$

$$= \tan^{-1}(e^x) + \frac{1}{2}\ln(e^{2x} + 1) + c.$$

7 *Solution*

Let $u = \sqrt{x}, x \geq 0$. Then $x = u^2, dx = 2udu$.

Hence

$$\int \frac{\sqrt{x}}{1 + x}dx = \int \frac{u}{1 + u^2}2udu = 2\int \frac{u^2}{1 + u^2}du = 2\int \frac{u^2 + 1 - 1}{1 + u^2}du = 2\int 1du - 2\int \frac{1}{1 + u^2}du$$

$$= 2u - 2\tan^{-1}(u) + c = 2\sqrt{x} - 2\tan^{-1}(\sqrt{x}) + c.$$

8 *Solution*

Let $x = \sin^2\theta$. Since $0 \leq x < 1$, we get $0 \leq \theta < \frac{\pi}{2}$, and $dx = 2\sin\theta\cos\theta d\theta, \theta = \sin^{-1}(\sqrt{x})$.

$$\int \sqrt{\frac{x}{1 - x}}dx = \int \sqrt{\frac{\sin^2\theta}{1 - \sin^2\theta}}2\sin\theta\cos\theta d\theta = 2\int \frac{\sin\theta}{\cos\theta}\sin\theta\cos\theta d\theta = 2\int \sin^2\theta d\theta$$

$$= 2\int \frac{1 - \cos 2\theta}{2}d\theta = \int 1d\theta - \int \cos 2\theta d\theta = \theta - \frac{1}{2}\sin 2\theta + c = \theta - \sin\theta\cos\theta + c$$

$$= \sin^{-1}(\sqrt{x}) - \sqrt{x(1 - x)} + c.$$

9 *Solution*

Let $x = 4\sin\theta$. Since $-4 \leq x \leq 4, (x \neq 0)$, we get $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, (\theta \neq 0)$, and

$dx = 4\cos\theta d\theta, \theta = \sin^{-1}\left(\frac{x}{4}\right)$.

Hence

$$\int_0^1 \frac{x}{x^4+1}dx = \int_0^1 \frac{\sqrt{u}}{u^2+1}\frac{1}{2\sqrt{u}}du = \frac{1}{2}\int_0^1 \frac{1}{u^2+1}du = \frac{1}{2}\left[\tan^{-1}u\right]_0^1 = \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{\pi}{8}.$$

14 *Solution*

Let $u = \sqrt{x}, 4 < x < 9 \Rightarrow 2 < u < 3$. Then $x = u^2, dx = 2udu$.

Hence

$$\int_4^9 \frac{1}{(x-1)\sqrt{x}}dx = \int_2^3 \frac{1}{(u^2-1)u}2udu = 2\int_2^3 \frac{1}{(u^2-1)}du = \int_2^3 \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \int_2^3 \frac{1}{u-1}du - \int_2^3 \frac{1}{u+1}du$$

$$= \left[\ln|u-1|\right]_2^3 - \left[\ln|u+1|\right]_2^3 = \ln 2 - \ln 1 - (\ln 4 - \ln 3) = \ln\frac{3}{2}.$$

15 *Solution*

Let $u^2 = 6 - x, u = \sqrt{6-x}, 2 < x < 6 \Rightarrow 0 < u < 2$. Then $x = 6 - u^2, dx = -2udu$.

Hence $\int_0^6 x\sqrt{6-x}dx = \int_0^2 (6-u^2)u(-2u)du = 2\int_0^2 (u^4 - 6u^2)du = 2\int_0^2 u^4 du - 12\int_0^2 u^2 du$

$$= \frac{2}{5}\left[u^5\right]_0^2 - 4\left[u^3\right]_0^2 = \frac{64}{5} - 32 = \frac{96}{5}.$$

16 *Solution*

Let $x = \tan\theta, \frac{1}{\sqrt{3}} < x < 1 \Rightarrow \frac{\pi}{6} < \theta < \frac{\pi}{4}, dx = \frac{1}{\cos^2\theta}d\theta$. Hence

$$\int_{1/\sqrt{3}}^1 \frac{1}{x^2\sqrt{1+x^2}}dx = \int_{\pi/6}^{\pi/4} \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}}\frac{1}{\cos^2\theta}d\theta = \int_{\pi/6}^{\pi/4} \frac{\cos\theta}{\sin^2\theta}d\theta = -\left[\frac{1}{\sin\theta}\right]_{\pi/6}^{\pi/4}$$

$$= -\frac{1}{\sin\pi/4} + \frac{1}{\sin\pi/6} = -\sqrt{2} + 2 = 2 - \sqrt{2}.$$

17 *Solution*

Let $x = \tan^2\theta, 0 < x < 1 \Rightarrow 0 < \theta < \frac{\pi}{4}, dx = \frac{2\tan\theta}{\cos^2\theta}d\theta$. Hence

$$\int_0^1 \frac{\sqrt{x}}{1+x}dx = \int_0^{\pi/4} \frac{\tan\theta}{1+\tan^2\theta}2\frac{\tan\theta}{\cos^2\theta}d\theta = 2\int_0^{\pi/4} \tan^2\theta d\theta = 2\int_0^{\pi/4} \frac{1-\cos^2\theta}{\cos^2\theta}d\theta$$

Hence $\int \frac{\sqrt{16-x^2}}{x^2}dx = \int \frac{\sqrt{16-16\sin^2\theta}}{16\sin^2\theta}4\cos\theta d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}d\theta$

$= \int \frac{d\theta}{\sin^2\theta} - \int d\theta = -\cot\theta - \theta + c = \frac{-\sqrt{1-x^2/16}}{x/4} - \sin^{-1}(x/4) + c$

10 *Solution*

Let $x = \cos 2\theta$. Since $-1 \le x < 1$, we get $-\frac{\pi}{4} \le \theta < \frac{\pi}{4}, \theta = \frac{1}{2}\cos^{-1}x, dx = -2\sin 2\theta d\theta$,

$2\cos^2\theta = 1+\cos 2\theta, 2\sin^2\theta = 1-\cos 2\theta$. Then we obtain

$\int \sqrt{\frac{1+x}{1-x}}dx = \int \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}(-2\sin 2\theta)d\theta = -4\int \frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta d\theta = -4\int \cos^2\theta d\theta$

$= -2\int(1+\cos 2\theta)d\theta = -\int 2d\theta - 2\int \cos 2\theta d\theta = -2\theta - \sin 2\theta + c = -\cos^{-1}x - \sqrt{1-x^2} + c$.

11 *Solution*

Let $t = \tan\frac{x}{2}, -\pi < x < \pi, x = 2\tan^{-1}t, dx = \frac{2}{1+t^2}dt$. Since $\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)}$, we obtain

$\int \operatorname{cosec} x dx = \int \frac{1}{\sin x}dx = \int \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}dx = \int \frac{1+t^2}{2t}\frac{2}{1+t^2}dt = \int \frac{1}{t}dt = \ln|t| + c = \ln\left|\tan(\frac{x}{2})\right| + c$.

12 *Solution*

Let $t = \tan\frac{x}{2}, -\pi < x < \pi,\ x = 2\tan^{-1}t,\ dx = \frac{2}{1+t^2}dt$. Since $\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$, we obtain

$\int \sec x dx = \int \frac{1}{\cos x}dx = \int \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}dx = \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}dt = 2\int \frac{1}{1-t^2}dt = \ln\left|\frac{1+t}{1-t}\right| + c$

$= \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + c = \ln\left|\tan(\frac{x}{2}+\frac{\pi}{4})\right| + c$.

13 *Solution*

Let $u = x^2$, and $0 < x < 1 \Rightarrow 0 < u < 1$. Then $x = \sqrt{u}, dx = \frac{1}{2\sqrt{u}}du$.

$$=2\int_0^{\pi/4}\frac{1}{\cos^2\theta}d\theta-2\int_0^{\pi/4}1d\theta=2[\tan\theta]_0^{\pi/4}-2[\theta]_0^{\pi/4}=2(\tan\pi/4-\tan 0)-\pi/2=2-\pi/2\,.$$

18 ***Solution***

Let $x=4\sin^2\theta, 0<x<2\Rightarrow 0<\theta<\dfrac{\pi}{4}, dx=8\sin\theta\cos\theta d\theta$. Hence

$$\int_0^2\sqrt{\frac{x}{4-x}}dx=\int_0^{\pi/4}\sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}8\sin\theta\cos\theta d\theta=8\int_0^{\pi/4}\sin^2\theta d\theta=4\int_0^{\pi/4}(1-\cos 2\theta)d\theta$$

$$=\pi-4\int_0^{\pi/4}\cos 2\theta d\theta=\pi-2[\sin 2\theta]_0^{\pi/4}=\pi-2\,.$$

19 ***Solution***

Let $t=\tan\dfrac{x}{2}, 0<x<\dfrac{\pi}{3}, 0<t<\dfrac{1}{\sqrt{3}}, x=2\tan^{-1}t, dx=\dfrac{2}{1+t^2}dt$. Since $\sin x=\dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$,

we obtain

$$\int_0^{\pi/3}\frac{1}{1-\sin x}dx=\int_0^{\pi/3}\frac{1}{1-\dfrac{2\tan(x/2)}{1+\tan^2(x/2)}}dx=\int_0^{1/\sqrt{3}}\frac{1}{1-\dfrac{2t}{1+t^2}}\frac{2}{1+t^2}dt=\int_0^{1/\sqrt{3}}\frac{1+t^2}{1+t^2-2t}\frac{2}{1+t^2}dt$$

$$=2\int_0^{1/\sqrt{3}}\frac{1}{(1-t)^2}dt=\left[\frac{2}{1-t}\right]_0^{1/\sqrt{3}}=\frac{2}{1-1/\sqrt{3}}-2=\frac{2\sqrt{3}}{\sqrt{3}-1}-2=\frac{2}{\sqrt{3}-1}=\sqrt{3}+1\,.$$

20 ***Solution***

Let $t=\tan\dfrac{x}{2},\ 0<x<\dfrac{\pi}{2}\Rightarrow 0<t<1,\ x=2\tan^{-1}t,\ dx=\dfrac{2}{1+t^2}dt$. Since $\cos x=\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$,

we obtain

$$\int_0^{\pi/2}\frac{1}{3+5\cos x}dx=\int_0^{\pi/2}\frac{1}{3+5\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}dx=\int_0^1\frac{1}{3+5\dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt=2\int_0^1\frac{1}{3+3t^2+5-5t^2}dt$$

$$=2\int_0^1\frac{1}{8-2t^2}dt=\int_0^1\frac{1}{4-t^2}dt=\frac{1}{4}\int_0^1\left(\frac{1}{2-t}+\frac{1}{2+t}\right)dt=\frac{1}{4}\int_0^1\frac{1}{2-t}dt+\frac{1}{4}\int_0^1\frac{1}{2+t}dt$$

$$=-\frac{1}{4}\left[\ln|2-t|\right]_0^1+\frac{1}{4}\left[\ln|2+t|\right]_0^1=-\frac{1}{4}(\ln 1-\ln 2)+\frac{1}{4}(\ln 3-\ln 2)=\frac{1}{4}\ln 3\,.$$

Exercise 5.4

1 *Solution*

Using the substitution $u=\sin x$, $\cos x dx = du$, we obtain

$$\int \sin^2 x\cos^3 x dx = \int \sin^2 x(1-\sin^2 x)\cos x dx = \int u^2(1-u^2)du = \int u^2 du - \int u^4 du = \frac{u^3}{3}-\frac{u^5}{5}+c$$

$$=\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+c.$$

2 *Solution*

Using the substitution $u=\cos x$, $\sin x dx = -du$, we obtain

$$\int \cos^2 x\sin^5 x dx = \int \cos^2 x(\sin^2 x)^2 \sin x dx = \int \cos^2 x(1-\cos^2 x)^2 \sin x dx$$

$$= \int \cos^2 x\sin x dx - 2\int \cos^4 x\sin x dx + \int \cos^6 x\sin x dx = -\int u^2 du + 2\int u^4 du - \int u^6 du$$

$$= -\frac{u^3}{3}+\frac{2}{5}u^5-\frac{u^7}{7}+c = -\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+c.$$

3 *Solution*

Using the substitution $u=\sin x$, $\cos x dx = du$, we obtain

$$\int \frac{\cos^3 x}{\sin^2 x}dx = \int \frac{1-\sin^2 x}{\sin^2 x}\cos x dx = \int \frac{1}{\sin^2 x}\cos x dx - \int \cos x dx = \int \frac{1}{u^2}du - \sin x + c$$

$$= -\frac{1}{\sin x} - \sin x + c = -\operatorname{cosec} x - \sin x + c.$$

4 *Solution*

Using the substitution $u=\cos x$, $\sin x dx = -du$, we obtain

$$\int \frac{\sin^3 x}{\cos^5 x}dx = \int \frac{1-\cos^2 x}{\cos^5 x}\sin x dx = \int \frac{1}{\cos^5 x}\sin x dx - \int \frac{1}{\cos^3 x}\sin x dx = -\int \frac{1}{u^5}du + \int \frac{1}{u^3}du$$

$$= \frac{1}{4u^4}-\frac{1}{2u^2}+c = \frac{1}{4\cos^4 x}-\frac{1}{2\cos^2 x}+c = \frac{1}{4}\sec^4 x-\frac{1}{2}\sec^2 x+c.$$

5 *Solution*

Using the substitution $u=\sin x$, $\cos x dx = du$, we have

$$\int \sqrt[3]{\sin x}\cos^3 xdx = \int \sqrt[3]{\sin x}(1-\sin^2 x)\cos xdx = \int \sqrt[3]{\sin x}\cos xdx - \int \sqrt[3]{\sin x}\sin^2 x\cos xdx$$

$$= \int u^{1/3}du - \int u^{1/3}u^2du = \frac{3}{4}u^{4/3} - \frac{3}{10}u^{10/3} + c = \frac{3}{4}\sqrt[3]{\sin^4 x} - \frac{3}{10}\sqrt[3]{\sin^{10} x} + c.$$

6 *Solution*

$$\int \cos 6x\cos 2xdx = \frac{1}{2}\int(\cos 4x + \cos 8x)dx = \frac{1}{2}\int \cos 4xdx + \frac{1}{2}\int \cos 8xdx$$

$$= \frac{1}{8}\sin 4x + \frac{1}{16}\sin 8x + c.$$

7 *Solution*

$$\int \sin 6x\sin 2xdx = \frac{1}{2}\int(\cos 4x - \cos 8x)dx = \frac{1}{2}\int \cos 4xdx - \frac{1}{2}\int \cos 8xdx$$

$$= \frac{1}{8}\sin 4x - \frac{1}{16}\sin 8x + c.$$

8 *Solution*

$$\int \sin 3x\cos xdx = \frac{1}{2}\int(\sin 2x + \sin 4x)dx = \frac{1}{2}\int \sin 2xdx + \frac{1}{2}\int \sin 4xdx$$

$$= -\frac{\cos 2x}{4} - \frac{\cos 4x}{8} + c.$$

9 *Solution*

$$\int \cos 3x\sin xdx = \frac{1}{2}\int(-\sin 2x + \sin 4x)dx = -\frac{1}{2}\int \sin 2xdx + \frac{1}{2}\int \sin 4xdx$$

$$= \frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + c.$$

10 *Solution*

$$\int \cos 4x\cos 2xdx = \frac{1}{2}\int(\cos 2x + \cos 6x)dx = \frac{1}{2}\int \cos 2xdx + \frac{1}{2}\int \cos 6xdx$$

$$= \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + c.$$

11 *Solution*

$$\int \sin 4x \cos 3x dx = \frac{1}{2}\int (\sin x + \sin 7x)dx = \frac{1}{2}\int \sin x dx + \frac{1}{2}\int \sin 7x dx$$

$$= -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + c\,.$$

12 *Solution*

$$\int \cos 5x \sin 2x dx = \frac{1}{2}\int (-\sin 3x + \sin 7x)dx = -\frac{1}{2}\int \sin 3x dx + \frac{1}{2}\int \sin 7x dx$$

$$= \frac{1}{6}\cos 3x - \frac{1}{14}\cos 7x + c\,.$$

13 *Solution*

$$\int_0^{\pi/4} (\tan^3 x + \tan x)dx = \int_0^{\pi/4} \tan x(\tan^2 x + 1)dx = \int_0^{\pi/4} \tan x \frac{1}{\cos^2 x} dx = \frac{1}{2}\left[\tan^2 x\right]_0^{\pi/4} = \frac{1}{2}.$$

14 *Solution*

$$\int_0^{\pi/3} (\sin^3 x - \sin x)dx = \int_0^{\pi/3} \sin x(\sin^2 x - 1)dx = -\int_0^{\pi/3} \cos^2 x \sin x dx = \left[\frac{\cos^3 x}{3}\right]_0^{\pi/3} = \frac{1}{24} - \frac{1}{3} = -\frac{7}{24}.$$

15 *Solution*

$$\int_0^{\pi/2} \sqrt{\cos x}\sin^3 x dx = \int_0^{\pi/2} \sqrt{\cos x}(1 - \cos^2 x)\sin x dx = \int_0^{\pi/2} \sqrt{\cos x}\sin x dx - \int_0^{\pi/2} \sqrt{\cos x}\cos^2 x \sin x dx$$

$$= -\left[2/3\cos^{3/2} x\right]_0^{\pi/2} + \frac{2}{7}\left[\cos^{7/2} x\right]_0^{\pi/2} = \frac{2}{3} - \frac{2}{7} = \frac{8}{21}.$$

16 *Solution*

$$\int_0^{\pi/4} \sin 4x \sin 2x dx = \frac{1}{2}\int_0^{\pi/4} (\sin 2x + \sin 6x)dx = \frac{1}{2}\int_0^{\pi/4} \sin 2x dx + \frac{1}{2}\int_0^{\pi/4} \sin 6x dx$$

$$= -\frac{1}{4}\left[\cos 2x\right]_0^{\pi/4} - \frac{1}{12}\left[\cos 6x\right]_0^{\pi/4} = \frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}.$$

Exercise 5.5

1 Solution

We note that $\frac{d}{dx}e^x = e^x$. Hence, using integration by parts, with x as the second function, removes x from the integrand, leaving e^x. Thus

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c = e^x(x-1) + c.$$

2 Solution

Since $(\ln x)' = \frac{1}{x}$, using integration by parts, with $\ln x$ as the second function, removes $\ln x$ from the integrand, leaving powers of x. Hence

$$\int x^2 \ln x dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x} dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$$

$$= \frac{x^3}{9}(3\ln x - 1) + c.$$

3 Solution

$$\int x \sin x dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + c.$$

4 Solution

Repeated application of integration by parts can be used to reduce the powers of x.

Hence

$$\int x^2 \cos x dx = x^2 \sin x - \int \sin x 2x dx = x^2 \sin x - 2(-x\cos x + \int \cos x dx)$$

$$= x^2 \sin x + 2x\cos x - 2\sin x + c.$$

5 Solution

$$\int x\cos^2 x dx = \int x\frac{1+\cos 2x}{2} dx = \frac{1}{2}\int x dx + \frac{1}{2}\int x\cos 2x dx = \frac{x^2}{4} + \frac{1}{2}\left(\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x dx\right)$$

$$=\frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+c\,.$$

6 ***Solution***

$$\int \tan^{-1} x dx = x\tan^{-1} x-\int\frac{x}{1+x^2}dx = x\tan^{-1} x-\frac{1}{2}\ln(x^2+1)+c = x\tan^{-1} x-\ln\sqrt{x^2+1}+c\,.$$

7 ***Solution***

$$\int x\tan^{-1} x dx=\frac{x^2}{2}\tan^{-1} x-\int\frac{x^2}{2}\frac{x}{1+x^2}dx=\frac{x^2}{2}\tan^{-1} x-\frac{1}{2}\int\frac{x^2+1-1}{1+x^2}dx$$
$$=\frac{x^2}{2}\tan^{-1} x-\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{1}{1+x^2}dx=\frac{x^2}{2}\tan^{-1} x-\frac{x}{2}+\frac{1}{2}\tan^{-1} x+c\,.$$

8 ***Solution***

$$\int e^x\cos x dx = e^x\cos x+\int e^x\sin x dx = e^x\cos x + e^x\sin x-\int e^x\cos x dx\,.$$

$2\int e^x\cos x dx=e^x(\cos x+\sin x)$. Hence $\int e^x\cos x dx=\frac{1}{2}e^x(\cos x+\sin x)+c$.

9 ***Solution***

We note that $\frac{d}{dx}\left(\frac{1}{\cos x}\right)=\frac{\sin x}{\cos^2 x}$. Hence, integration by parts, with x as the second functions, removes x from the integrand. Hence

$$\int x\sec x\tan x dx=\int\frac{x\sin x}{\cos^2 x}dx=\frac{x}{\cos x}-\int\frac{1}{\cos x}dx=\frac{x}{\cos x}-\ln|\sec x+\tan x|+c\,,$$

since $(\ln|\sec x+\tan x|)'=\frac{1}{\cos x}$.

10 ***Solution***

$$\int\frac{1}{\cos^3 x}dx=\int\frac{\cos^2 x+\sin^2 x}{\cos^3 x}dx=\int\frac{1}{\cos x}dx+\int\frac{\sin^2 x}{\cos^3 x}dx\,.$$

Furthermore $\int\frac{1}{\cos x}dx=\ln|\sec x+\tan x|+c$, and we have

$$\int \frac{\sin^2 x}{\cos^3 x}dx = \int \sin x \frac{\sin x}{\cos^3 x}dx = \frac{1}{2}\sin x \frac{1}{\cos^2 x} - \frac{1}{2}\int \frac{1}{\cos^2 x}\cos x dx = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\int \frac{1}{\cos x}dx$$

$= \dfrac{\sin x}{2\cos^2 x} - \dfrac{1}{2}\ln|\sec x + \tan x| + c$, since $\dfrac{d}{dx}\left(\dfrac{1}{\cos^2 x}\right) = \dfrac{2\sin x}{\cos^3 x}$. Finally,

$$\int \frac{1}{\cos^3 x}dx = \frac{1}{2}\ln|\sec x + \tan x| + \frac{\sin x}{2\cos^2 x} + c = \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\sec x \tan x + c.$$

11 *Solution*

$$\int_0^1 xe^{2x}dx = \frac{1}{2}\left[xe^{2x}\right]_0^1 - \frac{1}{2}\int_0^1 e^{2x}dx = \frac{1}{2}e^2 - \frac{1}{4}\left[e^{2x}\right]_0^1 = \frac{1}{4}e^2 + \frac{1}{4}.$$

12 *Solution*

$$\int_1^e (\ln x)^2 dx = \left[x\ln^2 x\right]_1^e - \int_1^e x(\ln^2 x)'dx = e - 2\int_1^e x\ln x \frac{1}{x}dx = e - 2\int_1^e \ln x dx =$$

$$= e - 2\left(\left[x\ln x - x\right]_1^e\right) = e - 2(e-(e-1)) = e - 2.$$

13 *Solution*

$$\int_0^{\pi/2} x\cos x dx = \left[x\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}\sin x dx = \frac{\pi}{2} + \left[\cos x\right]_0^{\pi/2} = \frac{\pi}{2} - 1.$$

14 *Solution*

$$\int_0^{\pi/2} x\sin x\cos x dx = \frac{1}{2}\int_0^{\pi/2} x\sin 2x dx = -\frac{1}{4}\left\{\left[x\cos 2x\right]_0^{\pi/2} - \int_0^{\pi/2}\cos 2x dx\right\}$$

$$= -\frac{1}{4}\left\{-\frac{\pi}{2} - \frac{1}{2}\left[\sin 2x\right]_0^{\pi/2}\right\} = \frac{\pi}{8}.$$

15 *Solution*

Let $n \geq 2$. Using the pattern $\int f^n(x)f'(x)dx = \dfrac{f^{n+1}(x)}{n+1} + c$, we get

$$I_n = \int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x\left(\frac{1}{\cos^2 x} - 1\right)dx$$

$= \displaystyle\int \tan^{n-2} x \frac{1}{\cos^2 x}dx - \int \tan^{n-2} x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$, where

$I_0 = \int dx = x + c$,

$I_1 = \int \tan x dx = -\ln|\cos x| + c$.

16 *Solution*

Let $n \geq 1$. Integration by parts, with $(\ln x)^n$ as the second function, reduces the power of $\ln x$.

$$I_n = \int x(\ln x)^n dx = \frac{x^2}{2}(\ln x)^n - \frac{1}{2}\int x^2 n(\ln x)^{n-1}\frac{1}{x}dx = \frac{x^2}{2}(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1}dx$$

$= \frac{x^2}{2}(\ln x)^n - \frac{n}{2}I_{n-1}$, where $I_0 = \int x dx = \frac{x^2}{2} + c$.

17 *Solution*

Let $n \geq 2$. Integration by parts, with $\sin^{n-1} x$ as the second function, reduces the power of $\sin x$.

$I_n = \int \sin^n x dx = -\cos x \sin^{n-1} x + \int \cos x(n-1)\sin^{n-2} x \cos x dx$

$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx - (n-1)\int \sin^n x dx$.

Hence

$I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n} I_{n-2}$,

where $I_0 = \int dx = x + c$, $\qquad I_1 = \int \sin x dx = -\cos x + c$.

Let for $n \geq 0$

$$I_n = \int_0^{\pi/2} \sin^n x dx.$$

Then for $n \geq 2$ we get $I_n = -\frac{1}{n}\left[\cos x \sin^{n-1} x\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} I_{n-2}$,

where $I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$, $\qquad I_1 = \int_0^{\pi/2} \sin x dx = -\left[\cos x\right]_0^{\pi/2} = 1$.

Thus $I_2 = \frac{1}{2} I_0 = \frac{\pi}{4}$, $\qquad I_3 = \frac{2}{3} I_1 = \frac{2}{3}$,

$I_4 = \frac{3}{4} I_2 = \frac{3}{16}\pi$, $\qquad I_5 = \frac{4}{5} I_3 = \frac{8}{15}$,

$I_6 = \frac{5}{6} I_4 = \frac{15}{96}\pi = \frac{5}{32}\pi$. Hence $I_5 \cdot I_6 = \frac{\pi}{12}$.

18 *Solution*

Let $n \geq 2$. Integration by parts yields

$$I_n = \int \sec^n dx = \int \frac{1}{\cos^n x} dx = \tan x \frac{1}{\cos^{n-2} x} - (n-2) \int \tan x \frac{1}{\cos^{n-1} x} \sin x dx$$

$$= \tan x \frac{1}{\cos^{n-2} x} - (n-2) \int \frac{1-\cos^2 x}{\cos^n x} dx$$

$$= \tan x \sec^{n-2} x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

$$= \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}.$$

Hence $I_n = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$,

where $I_0 = \int dx = x + c$, $\qquad I_1 = \int \dfrac{1}{\cos x} dx = \ln\left|\sec x + \tan x\right| + c$.

Let for $n \geq 0$

$$I_n = \int_0^{\pi/4} \sec^n x dx.$$

Then $\quad I_n = \dfrac{1}{n-1}\Big[\tan x \sec^{n-2} x\Big]_0^{\pi/4} + \dfrac{n-2}{n-1} I_{n-2} = \dfrac{(\sqrt{2})^{n-2}}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$,

where $I_0 = \displaystyle\int_0^{\pi/4} dx = \frac{\pi}{4}$, $\qquad I_1 = \displaystyle\int_0^{\pi/4} \sec x dx = \Big[\ln\left|\sec x + \tan x\right|\Big]_0^{\pi/4} = \ln(\sqrt{2}+1)$.

Thus we get

$$I_2 = 1, \qquad I_4 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}, \qquad I_6 = \frac{4}{5} + \frac{4}{5}\frac{4}{3} = \frac{28}{15}.$$

19 *Solution*

Repeated application of integration by parts, with x^n as the second function, removes powers of x from the integrand stepwise until the integral is known. Let $n \geq 2$, then

$$I_n = \int_0^{\pi/2} x^n \cos x dx = \Big[x^n \sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, n x^{n-1} dx$$

$$= \left(\frac{\pi}{2}\right)^n + n\left\{\Big[\cos x \, x^{n-1}\Big]_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2} \cos x dx\right\} = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2},$$

where

$$I_0 = \int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = 1,$$

$$I_1 = \int_0^{\pi/2} x\cos x dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx = \frac{\pi}{2} + [\cos x]_0^{\pi/2} = \frac{\pi}{2} - 1.$$

Thus we get

$$I_2 = \frac{\pi^2}{4} - 2, \qquad I_4 = \frac{\pi^4}{16} - 12\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^4}{16} - 3\pi^2 + 24,$$

$$I_6 = \frac{\pi^6}{64} - 30\left(\frac{\pi^4}{16} - 3\pi^2 + 24\right) = \frac{\pi^6}{64} - \frac{15}{8}\pi^4 + 90\pi^2 - 720.$$

20 *Solution*

Integration by parts, with $(1-x^3)^n$ as the second function, reduces the power of $(1-x^3)$.

Let $n \geq 1$, then

$$I_n = \int_0^1 x(1-x^3)^n dx = \frac{1}{2}\left[x^2(1-x^3)^n\right]_0^1 - \frac{1}{2}\int_0^1 x^2 n(1-x^3)^{n-1}(-3x^2)dx = \frac{3}{2}n\int_0^1 x(1-x^3)^{n-1}x^3 dx$$

$$= -\frac{3}{2}n\int_0^1 x(1-x^3)^{n-1}(1-x^3)dx + \frac{3}{2}n\int_0^1 x(1-x^3)^{n-1}dx = -\frac{3}{2}nI_n + \frac{3}{2}nI_{n-1}.$$

Hence $I_n = \dfrac{\frac{3}{2}n}{1+\frac{3}{2}n} I_{n-1} = \dfrac{3n}{2+3n} I_{n-1}$ with $I_0 = \int_0^1 x dx = \left[\dfrac{x^2}{2}\right]_0^1 = \dfrac{1}{2}$.

Furthermore, for $n \geq 0$ we have

$$I_n = \frac{3n}{2+3n} I_{n-1} = \frac{3n}{3n+2}\cdot\frac{3n-3}{3n-1} I_{n-2} = \frac{3n}{3n+2}\cdot\frac{3n-3}{3n-1}\cdot\frac{3n-6}{3n-4}\cdots\frac{6}{8}\cdot\frac{3}{5} I_0$$

$$= \frac{3^n n!}{(3n+2)(3n-1)\cdots 8\cdot 5\cdot 2}.$$

Exercise 5.6

1 *Solution*

(a) Let $x = a - u$, then $du = -dx, x = 0 \Rightarrow u = a, x = a \Rightarrow u = 0$, and

$$\int_0^a f(x)dx = -\int_a^0 f(a-u)du = \int_0^a f(a-u)du = \int_0^a f(a-x)dx.$$

(b)

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx = \int_0^\pi \frac{(\pi - x)\sin x}{1+\cos^2 x}dx = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx.$$

Hence

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx.$$

Using the substitution $u = \cos x, \sin x dx = -du, x = 0 \Rightarrow u = 1, x = \pi \Rightarrow u = -1$, we get

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = -\frac{\pi}{2}\int_1^{-1} \frac{1}{1+u^2}du = \pi\int_0^1 \frac{1}{1+u^2}du = \pi[\tan^{-1} u]_0^1 = \pi\frac{\pi}{4} = \frac{\pi^2}{4}.$$

2 *Solution*

(a) Using the following relations: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$,

$\sin\left(\frac{\pi}{2} - x\right) = \cos x, \cos^2 x = \frac{1+\cos 2x}{2}, \sin^2 x = \frac{1-\cos 2x}{2}$, we get

$$\int_0^{\pi/4} \frac{1-\sin 2x}{1+\sin 2x}dx = \int_0^{\pi/4} \frac{1-\sin[2(\pi/4 - x)]}{1+\sin[2(\pi/4 - x)]}dx = \int_0^{\pi/4} \frac{1-\cos 2x}{1+\cos 2x}dx = \int_0^{\pi/4} \frac{\sin^2 x}{\cos^2 x}dx$$

$$= \int_0^{\pi/4} \tan^2 x dx = \int_0^{\pi/4}\left(\frac{1}{\cos^2 x} - 1\right)dx = \int_0^{\pi/4} \frac{1}{\cos^2 x}dx - \int_0^{\pi/4} dx = [\tan x]_0^{\pi/4} - \frac{\pi}{4} = 1 - \frac{\pi}{4}.$$

(b) Using the relations, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$,

$$I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}dx = \int_0^{\pi/2} \frac{\cos^3(\pi/2 - x)}{\cos^3(\pi/2 - x) + \sin^3(\pi/2 - x)}dx = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}dx.$$

Hence

$$2I=\int_0^{\pi/2}\frac{\cos^3 x}{\cos^3 x+\sin^3 x}dx+\int_0^{\pi/2}\frac{\sin^3 x}{\sin^3 x+\cos^3 x}dx=\int_0^{\pi/2}\frac{\cos^3+\sin^3 x}{\cos^3 x+\sin^3 x}dx=\int_0^{\pi/2}1dx=\frac{\pi}{2}.$$

Thus $I=\frac{\pi}{4}$.

3 *Solution*

(a) Using the substitution $u=-x$, we have $\int_{-a}^{0}f(x)dx=-\int_{a}^{0}f(-u)du=\int_0^a f(-u)du=\int_0^a f(-x)dx$.

Then $\int_{-a}^{a}f(x)dx=\int_0^a f(x)dx+\int_{-a}^{0}f(x)dx=\int_0^a\{f(x)+f(-x)\}dx$.

(b)

$$\int_{-\pi/2}^{\pi/2}\frac{e^x}{1+e^x}\sin^2 xdx=\int_0^{\pi/2}\left\{\frac{e^x}{1+e^x}\sin^2 x+\frac{e^{-x}}{1+e^{-x}}\sin^2(-x)\right\}dx$$

$$=\int_0^{\pi/2}\left\{\frac{e^x}{1+e^x}\sin^2 x+\frac{1}{e^x+1}\sin^2 x\right\}dx=\int_0^{\pi/2}\frac{\sin^2 x}{1+e^x}(e^x+1)dx=\int_0^{\pi/2}\sin^2 xdx=\frac{1}{2}\int_0^{\pi/2}(1-\cos 2x)dx$$

$$=\frac{\pi}{4}-\frac{1}{2}\int_0^{\pi/2}\cos 2xdx=\frac{\pi}{4}-\frac{1}{4}[\sin 2x]_0^{\pi/2}=\frac{\pi}{4}.$$

4 *Solution*

(a) Using the relation $\int_{-a}^{a}f(x)dx=\int_0^a\{f(x)+f(-x)\}dx$, we get

$$\int_{-1}^{1}\frac{1}{1+e^{-x}}dx=\int_0^1\left\{\frac{1}{1+e^{-x}}+\frac{1}{1+e^x}\right\}dx=\int_0^1\left\{\frac{e^x}{e^x+1}+\frac{1}{1+e^x}\right\}dx=\int_0^1 1dx=1.$$

(b)

$$\int_{-\pi/4}^{\pi/4}\frac{1}{1+\sin x}dx=\int_0^{\pi/4}\left\{\frac{1}{1+\sin x}+\frac{1}{1+\sin(-x)}\right\}dx=\int_0^{\pi/4}\left\{\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right\}dx=\int_0^{\pi/4}\frac{2}{1-\sin^2 x}dx$$

$$=2\int_0^{\pi/4}\frac{1}{\cos^2 x}dx=2\left[\tan x\right]_0^{\pi/4}=2.$$

Diagnostic test 5

1 *Solution*

If $f(x)=\ln x$, then $f'(x)=\frac{1}{x}$, and the given integral follows the pattern

$\int f(x)f'(x)dx=\frac{1}{2}f^2(x)+c$. Hence $\int \frac{\ln x}{x}dx=\frac{1}{2}(\ln x)^2+c$.

2 *Solution*

If $f(x)=\frac{1}{x}$, then $f'(x)=-\frac{1}{x^2}$, and the given integral follows the pattern

$\int e^{f(x)}f'(x)dx=e^{f(x)}+c$. Hence $\int e^{1/x}\frac{1}{x^2}dx=-e^{1/x}+c$.

3 *Solution*

If $f(x)=\sin x+2$, then $f'(x)=\cos x$, and the given integral follows the pattern

$\int f^{-1}(x)f'(x)dx=\ln|f(x)|+c$. Hence $\int \frac{\cos x}{2+\sin x}dx=\ln|2+\sin x|+c=\ln(2+\sin x)+c$,

since $2+\sin x\geq 1$.

4 *Solution*

If $f(x)=x^2+1$, then $f'(x)=2x$, and using the pattern $\int \frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}+c$,

we get $\int\limits_0^2 \frac{x}{\sqrt{1+x^2}}dx=\left[\sqrt{1+x^2}\right]_0^2=\sqrt{5}-1$.

5 *Solution*

$\int \frac{x+1}{x^2+9}dx=\int \frac{x}{x^2+9}dx+\int \frac{1}{x^2+9}dx=\frac{1}{2}\ln(x^2+9)+\frac{1}{3}\tan^{-1}\frac{x}{3}+c$.

6 *Solution*

Let $\frac{x+7}{(x-1)(x+3)}\equiv\frac{a}{x-1}+\frac{b}{x+3}$, a, b, c constants.

Then $x+7 \equiv a(x+3)+b(x-1)$.

Put $x=1$: $8=4a \Rightarrow a=2$.

Put $x=-3$: $4=-4b \Rightarrow b=-1$.

Hence $\int \frac{x+7}{(x-1)(x+3)}dx = \int \left\{\frac{2}{x-1} - \frac{1}{x+3}\right\}dx = 2\int \frac{1}{x-1}dx - \int \frac{1}{x+3}dx = 2\ln|x-1| - \ln|x+3| + c$

$= \ln\left\{\frac{(x-1)^2}{|x+3|}\right\} + c$.

7 Solution

Let $\frac{2x^2-2x+1}{(x-2)(x^2+1)} \equiv \frac{a}{x-2} + \frac{bx+c}{x^2+1}$, $\quad a, b, c$ constants.

Then $2x^2-2x+1 \equiv a(x^2+1)+(bx+c)(x-2)$.

Put $x=2$: $5=5a \Rightarrow a=1$.

Equate coefficients of x^2: $2=a+b \Rightarrow b=1$.

Equate coefficients of x^1: $-2=-2b+c \Rightarrow c=0$.

Hence

$$\int \frac{2x^2-2x+1}{(x-2)(x^2+1)}dx = \int \left\{\frac{1}{x-2} + \frac{x}{x^2+1}\right\}dx = \int \frac{1}{x-2}dx + \int \frac{x}{x^2+1}dx = \ln|x-2| + \frac{1}{2}\ln(x^2+1) + c$$

$= \ln(|x-2|\sqrt{x^2+1}) + c$.

8 Solution

$$\int_2^4 \frac{(x^2-1)^2}{x}dx = \int_2^4 \frac{x^4-2x^2+1}{x}dx = \int_2^4 (x^3-2x+1/x)dx = \int_2^4 x^3dx - 2\int_2^4 xdx + \int_2^4 \frac{1}{x}dx$$

$$= \left[\frac{x^4}{4}\right]_2^4 - \left[x^2\right]_2^4 + \left[\ln x\right]_2^4 = 64-4-16+4+\ln 4 - \ln 2 = 48 + \ln 2.$$

9 Solution

Using the substitution $u=\sqrt{x}$, $x=u^2$, $dx=2udu$, we have

$$\int \frac{1}{(1+x)\sqrt{x}}dx = \int \frac{2u}{(1+u^2)u}du = 2\int \frac{1}{1+u^2}du = 2\tan^{-1}u + c = 2\tan^{-1}(\sqrt{x}) + c.$$

10 Solution

Using the substitution $u^2 = x+1,\ x = u^2 - 1,\ dx = 2udu$, we have

$$\int \frac{x}{\sqrt{x+1}}dx = \int \frac{u^2-1}{u}\cdot 2udu = 2\int (u^2-1)du = \frac{2}{3}u^3 - 2u + c = \frac{2}{3}(x+1)^{3/2} - 2\sqrt{x+1} + c.$$

11 Solution

Using the substitution $x = \sec\theta,\ dx = \dfrac{\sin\theta}{\cos^2\theta}d\theta,\ x = 2 \Rightarrow \theta = \dfrac{\pi}{3},\ x = \sqrt{2} \Rightarrow \theta = \dfrac{\pi}{4}$, we have

$$\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}dx = \int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sqrt{(1/\cos^2\theta)-1}}\cdot\frac{\sin\theta}{\cos^2\theta}d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{\tan\theta}\cdot\tan\theta d\theta = [\theta]_{\pi/4}^{\pi/3} = \pi/12.$$

12 Solution

Using the substitution

$t = \tan\dfrac{x}{2},\ \cos x = \dfrac{1-t^2}{1+t^2},\ x = 0 \Rightarrow t = 0,\ x = \dfrac{\pi}{2} \Rightarrow t = 1,\ x = 2\tan^{-1}t,\ dx = \dfrac{2}{1+t^2}dt$, we get

$$\int_0^{\pi/2} \frac{1}{1+\cos x}dx = \int_0^1 \frac{1}{1+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt = \int_0^1 1dt = 1.$$

13 Solution

Using the substitution $u = \sin x,\ du = \cos x dx$,

$$\int \sqrt{\sin x}\cos^3 x dx = \int \sqrt{\sin x}(1-\sin^2 x)\cos x dx = \int \sqrt{u}(1-u^2)du = \int \sqrt{u}du - \int u^{5/2}du$$

$$= \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2} + c = \frac{2}{3}\sin^{3/2}x - \frac{2}{7}\sin^{7/2}x + c.$$

14 Solution

Using the formula $2\sin p\cos q = \sin(p-q) + \sin(p+q)$, we get

$$\int \sin 4x\cos x dx = \frac{1}{2}\int \sin 3x dx + \frac{1}{2}\int \sin 5x dx = -\frac{1}{6}\cos 3x - \frac{1}{10}\cos 5x + c.$$

15 *Solution*

Integration by parts, with x^2 as the second function, removes powers of x from the integrand

$$\int x^2 e^x dx = e^x x^2 - 2\int xe^x dx = e^x x^2 - 2\left\{xe^x - \int e^x dx\right\} = x^2 e^x - 2\left\{xe^x - e^x\right\} + c$$

$$= x^2 e^x - 2xe^x + 2e^x + c.$$

16 *Solution*

Integration by parts, with x as the second function, removes powers of x from the integrand

$$\int x\cos 2x dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c.$$

17 *Solution*

$$\int_0^1 \tan^{-1} x dx = \left[x\tan^{-1} x\right]_0^1 - \int_0^1 x(\tan^{-1} x)' dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2}\left[\ln(x^2+1)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

18 *Solution*

Let $n \geq 1$, and

$$I_n = \int_1^e (\ln x)^n dx = \left[x(\ln x)^n\right]_1^e - \int_1^e x\left((\ln x)^n\right)' dx = e - \int_1^e xn(\ln x)^{n-1}\frac{1}{x} dx = e - n\int_1^e (\ln x)^{n-1} dx$$

$$= e - nI_{n-1},$$

where $I_0 = \int_1^e 1 dx = e - 1$.

Hence $I_4 = e - 4I_3 = e - 4(e - 3I_2) = -3e + 12(e - 2I_1) = 9e - 24(e - I_0) = -15e + 24(e-1)$

$= 9e - 24$.

19 *Solution*

It is clear that $f(x) = x^6 \sin x$ is an odd function, since

$f(-x) = (-x)^6 \sin(-x) = -x^6 \sin x = f(x)$. Hence

$$\int_{-\pi/2}^{\pi/2} x^6 \sin x dx = 0,$$

because of the fact that

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} \{f(x) + f(-x)\}dx = 0 \text{ if } f(x) \text{ is odd.}$$

20 *Solution*

Using the substitution $u = a - x,\, du = -dx,\, x = 0 \Rightarrow u = a,\, x = a \Rightarrow u = 0$, we get

$$\int_{0}^{a} f(x)dx = -\int_{a}^{0} f(a-u)du = \int_{0}^{a} f(a-u)du = \int_{0}^{a} f(a-x)dx.$$

Hence

$$I = \int_{0}^{\pi/2} \frac{\cos x}{\cos x + \sin x}dx = \int_{0}^{\pi/2} \frac{\cos(\pi/2 - x)}{\cos(\pi/2 - x) + \sin(\pi/2 - x)}dx = \int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx,$$

since $\cos(\pi/2 - x) = \sin x, \sin(\pi/2 - x) = \cos x$. Then

$$2I = \int_{0}^{\pi/2} \frac{\cos x}{\cos x + \sin x}dx + \int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx = \int_{0}^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x}dx = \int_{0}^{\pi/2} 1dx = \frac{\pi}{2}.$$

Hence $I = \dfrac{\pi}{4}$.

Further questions 5

1 *Solution*

Using the pattern $\int f(x)f'(x)dx = \frac{1}{2}f^2(x)+c$ with $f(x)=\tan^{-1}x$, we have

$$\int \frac{\tan^{-1}x}{1+x^2}dx = \frac{1}{2}(\tan^{-1}x)^2+c.$$

2 *Solution*

Using the pattern $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+c$ with $u=x+2, a=1$, we get

$$\int \frac{1}{x^2+4x+3}dx = \int \frac{1}{(x+2)^2-1}dx = \int \frac{1}{u^2-1}du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|+c = \frac{1}{2}\ln\left|\frac{x+1}{x+3}\right|+c.$$

3 *Solution*

Using the substitution $e^x+1=u, du=e^x dx$, we have

$$\int \frac{1}{1+e^{-x}}dx = \int \frac{e^x}{e^x+1}dx = \int \frac{1}{u}du = \ln|u|+c = \ln(e^x+1)+c.$$

4 *Solution*

Using the pattern $\int \frac{1}{u^2+a^2}du = \frac{1}{a}\tan^{-1}\frac{u}{a}+c$ with $u=x+2, a=1$, we get

$$\int \frac{1}{x^2+4x+5}dx = \int \frac{1}{(x+2)^2+1}dx = \int \frac{1}{u^2+1}du = \tan^{-1}u+c = \tan^{-1}(x+2)+c.$$

5 *Solution*

Integration by parts leads to a more simple integral

$$\int \ln(x^2-1)dx = x\ln(x^2-1) - \int x(\ln(x^2-1))'dx = x\ln(x^2-1) - \int \frac{x}{x^2-1}2xdx$$

$$= x\ln(x^2-1) - 2\int \frac{x^2-1+1}{x^2-1}dx = x\ln(x^2-1) - 2\int dx - 2\int \frac{1}{x^2-1}dx = x\ln(x^2-1) - 2x - \ln\left|\frac{x-1}{x+1}\right|+c.$$

6 *Solution*

$$\int \frac{1+\sin x}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx + \int \frac{\sin x}{\cos^2 x}dx = \tan x + \frac{1}{\cos x}+c = \tan x + \sec x + c.$$

7 Solution

$$\int \sin 5x \cos 3x dx = \frac{1}{2}\int (\sin 2x + \sin 8x) dx = \frac{1}{2}\int \sin 2x dx + \frac{1}{2}\int \sin 8x dx = -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c\,.$$

8 Solution

Using the substitution $x-1=u$ and the pattern $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c$, we have

$$\int \frac{1}{3+2x-x^2}dx = \int \frac{1}{4-(x-1)^2}dx = -\int \frac{1}{(x-1)^2-4}dx = -\int \frac{1}{u^2-4}du = -\frac{1}{4}\ln\left|\frac{u-2}{u+2}\right| + c$$
$$= \frac{1}{4}\ln\left|\frac{x+1}{x-3}\right| + c.$$

9 Solution

Using the substitution $e^x = u, du = e^x dx$, we get

$$\int \frac{1}{e^x + e^{-x}}dx = \int \frac{e^x}{e^{2x}+1}dx = \int \frac{1}{u^2+1}du = \tan^{-1} u + c = \tan^{-1} e^x + c\,.$$

10 Solution

Integration by parts leads to a more simple integral

$$\int \ln(x^2+1)dx = x\ln(x^2+1) - \int x(\ln(x^2+1))'dx = x\ln(x^2+1) - \int \frac{x}{x^2+1}2xdx$$
$$= x\ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}dx = x\ln(x^2+1) - 2\int dx + 2\int \frac{1}{x^2+1}dx = x\ln(x^2+1) - 2x + 2\tan^{-1} x + c\,.$$

11 Solution

$$\int (\tan x + \cot x)dx = \int \frac{\sin x}{\cos x}dx + \int \frac{\cos x}{\sin x}dx = -\ln|\cos x| + \ln|\sin x| + c = \ln|\tan x| + c\,.$$

12 Solution

Using the substitution $u = x-1$ and the pattern $\int \frac{1}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$ with $a=2$, we get

$$\int \frac{1}{\sqrt{3+2x-x^2}}dx = \int \frac{1}{\sqrt{4-(x-1)^2}}dx = \int \frac{1}{\sqrt{4-u^2}}du = \sin^{-1}\frac{u}{2} + c = \sin^{-1}\left(\frac{x-1}{2}\right) + c\,.$$

13 *Solution*

$$\int \sin 4x \sin 2x dx = \frac{1}{2}\int (\cos 2x - \cos 6x) dx = \frac{1}{2}\int \cos 2x dx - \frac{1}{2}\int \cos 6x dx = \frac{1}{4}\sin 2x - \frac{1}{12}\sin 6x + c.$$

14 *Solution*

$$\int \frac{x^2}{x^2-1}dx = \int \frac{x^2-1+1}{x^2-1}dx = \int 1 dx + \int \frac{1}{x^2-1}dx = x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + c = x - \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + c.$$

15 *Solution*

Using the substitution $x = 2\sin u, dx = 2\cos u du$, we get

$$\int \sqrt{4-x^2}dx = \int \sqrt{4(1-\sin^2 u)}2\cos u du = 4\int \cos^2 u du = 2\int (1+\cos 2u) du = 2u + \sin 2u + c$$

$$= 2\sin^{-1}\frac{x}{2} + 2\sin u \cos u + c = 2\sin^{-1}\frac{x}{2} + x\sqrt{1-\frac{x^2}{4}} + c = 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + c.$$

16 *Solution*

Using the substitution $u = \ln x, du = \frac{1}{x}dx$, we get

$$\int \frac{1}{x}\sec^2(\ln x)dx = \int \sec^2 u du = \int \frac{1}{\cos^2 u}du = \tan u + c = \tan(\ln x) + c.$$

17 *Solution*

Using the substitution $t = \tan\frac{x}{2}, \cos x = \frac{1-t^2}{1+t^2}, x = 2\tan^{-1} t, dx = \frac{2}{1+t^2}dt$, we have

$$\int \frac{1}{1-\cos x}dx = \int \frac{1}{1-(1-t^2)/(1+t^2)}\frac{2}{1+t^2}dt = 2\int \frac{1}{2t^2}dt = \int \frac{1}{t^2}dt = -\frac{1}{t} + c = -\frac{1}{\tan x/2} + c$$

$$= -\cot\frac{x}{2} + c.$$

18 *Solution*

Make the substitution $u = x+1$. Hence

$$\int \frac{2x+1}{x^2+2x+2}dx = \int \frac{2(x+1)-1}{(x+1)^2+1}dx = \int \frac{2u-1}{u^2+1}du = 2\int \frac{u}{u^2+1}du - \int \frac{1}{u^2+1}du$$

$$= \ln(u^2+1) - \tan^{-1} u + c = \ln(x^2+2x+2) - \tan^{-1}(x+1) + c.$$

19 *Solution*

Make the substitution $\cos x = u,\ du = -\sin x dx$. Then we have

$$\int \sin^3 x \cos^2 x dx = \int (\cos^2 x - 1)\cos^2 x(-\sin x)dx = \int (u^2 - 1)u^2 du = \int u^4 du - \int u^2 du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c.$$

20 *Solution*

Integration by parts leads to the following relation

$$I = \int \sqrt{16+x^2}dx = x\sqrt{16+x^2} - \int x\left(\sqrt{16+x^2}\right)' dx = x\sqrt{16+x^2} - \int \frac{x^2}{\sqrt{16+x^2}}dx$$

$$= x\sqrt{16+x^2} - \int \frac{x^2+16-16}{\sqrt{16+x^2}}dx = x\sqrt{16+x^2} - \int \sqrt{x^2+16}dx + 16\int \frac{1}{\sqrt{16+x^2}}dx$$

$$= x\sqrt{16+x^2} - I + 16\int \frac{1}{\sqrt{16+x^2}}dx.$$

Hence

$$\int \sqrt{16+x^2}dx = \frac{1}{2}x\sqrt{16+x^2} + 8\int \frac{1}{\sqrt{16+x^2}}dx = \frac{x}{2}\sqrt{16+x^2} + 8\ln(x+\sqrt{16+x^2}) + c.$$

21 *Solution*

Using the pattern $\int e^{f(x)} f'(x)dx = e^{f(x)} + c$, we get $\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx = e^{\sin^{-1}x} + c$.

22 *Solution*

Let $\frac{3x^2-6x+1}{(x-3)(x^2+1)} \equiv \frac{a}{x-3} + \frac{bx+c}{x^2+1}$, a, b, c constants.

Then $3x^2 - 6x + 1 \equiv a(x^2+1) + (bx+c)(x-3)$.

Put $x = 3$: $10 = 10a \Rightarrow a = 1$.

Equate coefficients of x^2: $3 = a + b \Rightarrow b = 2$.

Equate constant terms: $1 = a - 3c \Rightarrow c = 0$.

Hence

$$\int \frac{3x^2-6x+1}{(x-3)(x^2+1)}dx = \int \left\{ \frac{1}{x-3} + \frac{2x}{x^2+1} \right\} dx = \int \frac{1}{x-3}dx + 2\int \frac{x}{x^2+1}dx = \ln|x-3| + \ln(x^2+1) + c$$
$$= \ln\left(|x-3| \cdot (x^2+1)\right) + c.$$

23 Solution

Make the substitution $x-1=u^2$, $dx = 2udu$ .Then

$$\int \frac{1}{x^2\sqrt{x-1}}dx = \int \frac{1}{(u^2+1)^2 u} 2udu = 2\int \frac{1}{(u^2+1)^2}du = 2\int \frac{1+u^2-u^2}{(u^2+1)^2}du$$
$$= 2\int \frac{1}{u^2+1}du - 2\int \frac{u^2}{(u^2+1)^2}du = 2\tan^{-1}u + \int u\left(\frac{1}{u^2+1}\right)' u = 2\tan^{-1}u + \frac{u}{u^2+1} - \int \frac{1}{u^2+1}du$$
$$= 2\tan^{-1}u + \frac{u}{u^2+1} - \tan^{-1}u + c = \tan^{-1}u + \frac{u}{u^2+1} + c = \tan^{-1}(\sqrt{x-1}) + \frac{\sqrt{x-1}}{x} + c.$$

24 Solution

Let $\dfrac{2x^2-x+20}{(x-2)(x^2+9)} \equiv \dfrac{a}{x-2} + \dfrac{bx+c}{x^2+9}$, a, b, c constants.

Then $2x^2-x+20 \equiv a(x^2+9)+(bx+c)(x-2)$.

Put $x=2$: $26 = 13a \Rightarrow a = 2$.

Equate coefficients of x^2: $2 = a+b \Rightarrow b = 0$.

Equate constant terms: $20 = 9a - 2c \Rightarrow c = -1$.

Hence

$$\int \frac{2x^2-x+20}{(x-2)(x^2+9)}dx = \int \left\{ \frac{2}{x-2} - \frac{1}{x^2+9} \right\} dx = 2\int \frac{1}{x-2}dx + \int \frac{1}{x^2+9}dx$$
$$= 2\ln|x-2| - \frac{1}{3}\tan^{-1}\frac{x}{3} + c.$$

25 Solution

Make the substitution $e^x = u$, $du = e^x dx$ .Then

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}u + c = \sin^{-1}(e^x) + c.$$

26 *Solution*

Let $\dfrac{12}{(x^2+4)(x^2+16)} \equiv \dfrac{ax+b}{x^2+4}+\dfrac{cx+d}{x^2+16}$, $\quad a, b, c$ constants.

Then $12 \equiv (ax+b)(x^2+16)+(cx+d)(x^2+4)$.

Equate coefficients of x^3: $\quad 0 = a+c$.

Equate coefficients of x^2: $\quad 0 = b+d$.

Equate coefficients of x^1: $\quad 0 = 16a+4c$.

Equate constant terms: $\quad 12 = 16b+4d$.

Thus $a = c = 0$, $b = 1$, $d = -1$, and

$$\int \frac{12}{(x^2+4)(x^2+16)}dx = \int\left(\frac{1}{x^2+4}-\frac{1}{x^2+16}\right)dx = \int\frac{1}{x^2+4}dx - \int\frac{1}{x^2+16}dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c.$$

27 *Solution*

Make the substitution $\cos x = u, du = -\sin x dx$ .Then we have

$$\int\frac{\sin^3 x}{\cos^2 x}dx = \int\frac{(\cos^2 x-1)}{\cos^2 x}(-\sin x)dx = \int\frac{u^2-1}{u^2}du = \int 1du - \int\frac{1}{u^2}du = u+\frac{1}{u}+c$$

$$= \cos x + \frac{1}{\cos x}+c = \cos x + \operatorname{cosec} x + c.$$

28 *Solution*

Make the substitution $t = \tan x, x = \tan^{-1} t, x = 0 \Rightarrow t = 0, x = \dfrac{\pi}{4} \Rightarrow t = 1, dx = \dfrac{1}{1+t^2}dt$.

Then $\displaystyle\int_0^{\pi/4}\frac{1-\tan x}{1+\tan x}dx = \int_0^1\frac{1-t}{1+t}\cdot\frac{1}{1+t^2}dt$.

Let

$\dfrac{1-t}{1+t}\cdot\dfrac{1}{1+t^2} \equiv \dfrac{a}{1+t}+\dfrac{bt+c}{t^2+1}$, $\quad a, b, c$ constant.

Then $1-t \equiv a(t^2+1)+(bt+c)(t+1)$.

Put $x = -1$: $\quad 2 = 2a \Rightarrow a = 1$.

Equate coefficients of t^2: $\quad 0 = a+b \Rightarrow b = -1$.

Equate constant terms: $1 = a + c \Rightarrow c = 0$.

Hence

$$\int_0^{\pi/4} \frac{1-\tan x}{1+\tan x}dx = \int_0^1 \left\{\frac{1}{1+t} - \frac{t}{1+t^2}\right\}dt = \int_0^1 \frac{1}{1+t}dt - \int_0^1 \frac{t}{1+t^2}dt = \left[\ln(t+1)\right]_0^1 - \frac{1}{2}\left[\ln(t^2+1)\right]_0^1$$

$$= \ln 2 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln 2.$$

29 *Solution*

$$\int_0^1 \frac{x+1}{x^2+1}dx = \int_0^1 \frac{x}{x^2+1}dx + \int_0^1 \frac{1}{x^2+1}dx = \frac{1}{2}\left[\ln(x^2+1)\right]_0^1 + \left[\tan^{-1}x\right]_0^1 = \frac{1}{2}\ln 2 + \frac{\pi}{4}.$$

30 *Solution*

It is clear that

$$I = \int_0^{\pi/2} \sqrt{1+\sin 2x}\,dx = \frac{1}{2}\int_0^{\pi} \sqrt{1+\sin x}\,dx = \int_0^{\pi/2} \sqrt{1+\sin x}\,dx.$$

Make the substitution $\sin x = u, x = \sin^{-1}u, dx = \frac{1}{\sqrt{1-u^2}}du, x = 0 \Rightarrow u = 0, x = \frac{\pi}{2} \Rightarrow u = 1$.

Hence $I = \int_0^1 \frac{\sqrt{1+u}}{\sqrt{1-u^2}}du = \int_0^1 \frac{1}{\sqrt{1-u}}du = -2\left[\sqrt{1-u}\right]_0^1 = 2$.

31 *Solution*

Let $\frac{5x^2+4x-20}{(x+2)(x^2+4)} \equiv \frac{a}{x+2} + \frac{bx+c}{x^2+4}$, a, b, c constants.

Then $5x^2+4x-20 \equiv a(x^2+4)+(bx+c)(x+2)$.

Put $x = -2$: $-8 = 8a \Rightarrow a = -1$.

Equate coefficients of $x^2 : 5 = a + b \Rightarrow b = 6$.

Equate constant terms: $-20 = 4a + 2c \Rightarrow c = -8$.

Hence

$$\int_0^2 \frac{5x^2+4x-20}{(x+2)(x^2+4)}dx = \int_0^2 \left\{\frac{-1}{x+2} + \frac{6x-8}{x^2+4}\right\}dx = -\int_0^2 \frac{1}{x+2}dx + 6\int_0^2 \frac{x}{x^2+4}dx - 8\int_0^2 \frac{1}{x^2+4}dx$$

$$= -\left[\ln(x+2)\right]_0^2 + 3\left[\ln(x^2+4)\right]_0^2 - 4\left[\tan^{-1}\frac{x}{2}\right]_0^2 = -\ln 4 + \ln 2 + 3\ln 8 - 3\ln 4 - \pi = 2\ln 2 - \pi.$$

32 *Solution*

Using the substitution $t=\tan\frac{x}{2}$, and

$\sin x=\frac{2t}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$, $dx=\frac{2}{1+t^2}dt$, $x=0\Rightarrow t=0$, $x=\frac{\pi}{2}\Rightarrow t=1$, we have

$$\int_0^{\pi/2}\frac{1}{3\cos x+4\sin x+5}dx=\int_0^1\left[3\frac{1-t^2}{1+t^2}+4\frac{2t}{1+t^2}+5\right]^{-1}\frac{2}{1+t^2}dt=2\int_0^1\frac{1}{3-3t^2+8t+5+5t^2}dt$$

$$=2\int_0^1\frac{1}{2t^2+8t+8}dt=\int_0^1\frac{1}{(t+2)^2}dt=-\left[\frac{1}{t+2}\right]_0^1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}.$$

33 *Solution*

Let $\frac{3x^2-ax}{(x-2a)(x^2+a^2)}\equiv\frac{b}{x-2a}+\frac{cx+d}{x^2+a^2}$.

Then $3x^2-ax\equiv b(x^2+a^2)+(cx+d)(x-2a)$.

Put $x=2a$: $\quad 10a^2=5a^2b\Rightarrow b=2$.

Equate coefficients of $x^2: 3=b+c\Rightarrow c=1$. Equate constant terms: $0=ba^2-2ad\Rightarrow d=a$.

Hence

$$\int_0^a\frac{3x^2-ax}{(x-2a)(x^2+a^2)}dx=\int_0^a\left\{\frac{2}{x-2a}+\frac{x+a}{x^2+a^2}\right\}dx$$

$$=2\int_0^a\frac{1}{x-2a}dx+\int_0^a\frac{x}{x^2+a^2}dx+a\int_0^a\frac{1}{x^2+a^2}dx=2\big[\ln|x-2a|\big]_0^a+\frac{1}{2}\left[\ln(x^2+a^2)\right]_0^a+\left[\tan^{-1}x/a\right]_0^a$$

$$=-2\ln 2+\frac{1}{2}\ln 2+\frac{\pi}{4}=\frac{\pi}{4}-\frac{3}{2}\ln 2.$$

34 *Solution*

Let n be positive integer. Then $\int_{\pi/2}^{\pi}\cos nx dx=\frac{1}{n}[\sin nx]_{\pi/2}^{\pi}=\frac{1}{n}(\sin\pi n-\sin\frac{\pi}{2}n)$

$$=-\frac{1}{n}\sin\frac{\pi}{2}n=\begin{Bmatrix}0, \text{ if } n=2m+2,\\ -\frac{1}{n}, \text{ if } n=4m+1,\\ \frac{1}{n}, \text{ if } n=4m+3,\end{Bmatrix}\qquad \text{where } m=0,1,2,3,\text{K}$$

35 *Solution*

Since $2\cos p\cos q=\cos(p-q)+\cos(p+q)$,

we get $\cos mx\cos nx=\frac{1}{2}\{\cos(m-n)x+\cos(m+n)x\}$.

Hence $I=\int_0^{2\pi}\cos mx\cos nx dx=\frac{1}{2}\int_0^{2\pi}\cos(m-n)xdx+\frac{1}{2}\int_0^{2\pi}\cos(m+n)xdx$.

Let $m\neq n$, then $I=\frac{1}{2(m-n)}[\sin(m-n)x]_0^{2\pi}+\frac{1}{2(m+n)}[\sin(m+n)x]_0^{2\pi}=0$.

Let $m=n$, then $I=\frac{1}{2}2\pi+\frac{1}{4m}[\sin 2mx]_0^{2\pi}=\pi$, since $\int_0^{2\pi}dx=2\pi$.

36 *Solution*

Let us show that $\frac{1}{9-8\sin^2 x}=\frac{\sec^2 x}{9+\tan^2 x}$. Since $\sin^2 x=\frac{\tan^2 x}{1+\tan^2 x}$, we get

$$\frac{1}{9-8\sin^2 x}=\left[9-8\frac{\tan^2 x}{1+\tan^2 x}\right]^{-1}=\left[\frac{9+9\tan^2 x-8\tan^2 x}{1+\tan^2 x}\right]^{-1}=\frac{1+\tan^2 x}{9+\tan^2 x}=\frac{\sec^2 x}{9+\tan^2 x}.$$

Hence, using the substitution $u=\tan x, du=\sec^2 xdx, x=0\Rightarrow u=0, x=\frac{\pi}{3}\Rightarrow u=\sqrt{3}$,

we have $\int_0^{\pi/3}\frac{1}{9-8\sin^2 x}dx=\int_0^{\pi/3}\frac{\sec^2 x}{9+\tan^2 x}dx=\int_0^{\sqrt{3}}\frac{1}{9+u^2}du=\frac{1}{3}\left[\tan^{-1}\frac{u}{3}\right]_0^{\sqrt{3}}=\frac{1}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{18}$.

37 *Solution*

Let us show that $\frac{1}{9-10\sin^2 x}=\frac{\sec^2 x}{9-\tan^2 x}$. Since $\sin^2 x=\frac{\tan^2 x}{1+\tan^2 x}$, we get

$$\frac{1}{9-10\sin^2 x}=\left[9-10\frac{\tan^2 x}{1+\tan^2 x}\right]^{-1}=\left[\frac{9-\tan^2 x}{1+\tan^2 x}\right]^{-1}=\frac{1+\tan^2 x}{9-\tan^2 x}=\frac{\sec^2 x}{9-\tan^2 x}.$$

Hence, using the substitution $u=\tan x, du=\sec^2 xdx, x=0\Rightarrow u=0, x=\frac{\pi}{3}\Rightarrow u=\sqrt{3}$,

we have

$$\int_0^{\pi/3} \frac{1}{9-10\sin^2 x}dx = \int_0^{\pi/3} \frac{\sec^2 x}{9-\tan^2 x}dx = \int_0^{\sqrt{3}} \frac{1}{9-u^2}du = \frac{1}{6}\left[\ln\left|\frac{3+u}{3-u}\right|\right]_0^{\sqrt{3}} = \frac{1}{6}\ln\frac{3+\sqrt{3}}{3-\sqrt{3}}$$

$$= \frac{1}{6}\ln\frac{(3+\sqrt{3})^2}{3^2-(\sqrt{3})^2} = \frac{1}{6}\ln(2+\sqrt{3}).$$

38 *Solution*

Make the substitution. $x = 5\sin^2\theta + \cos^2\theta = 4\sin^2\theta + 1$. Then $dx = 8\sin\theta\cos\theta d\theta$,

$x = 2 \Rightarrow \theta = \frac{\pi}{2}, x = 4 \Rightarrow \theta = \frac{\pi}{3}$,

since $x = 4\sin^2\theta + 1$. Hence $\int_2^4 \sqrt{\frac{5-x}{x-1}} = \int_{\pi/6}^{\pi/3} \sqrt{\frac{5-5\sin^2\theta-\cos^2\theta}{5\sin^2\theta+\cos^2\theta-1}}8\sin\theta\cos\theta d\theta$

$$= 8\int_{\pi/6}^{\pi/3} \sqrt{\frac{4\cos^2\theta}{4\sin^2\theta}}\sin\theta\cos\theta d\theta = 8\int_{\pi/6}^{\pi/3} \cos^2\theta d\theta = 4\int_{\pi/6}^{\pi/3} (1+\cos 2\theta)d\theta$$

$$= \frac{2}{3}\pi + 4\int_{\pi/6}^{\pi/3} \cos 2\theta d\theta = \frac{2}{3}\pi + 2[\sin 2\theta]_{\pi/6}^{\pi/3} = \frac{2}{3}\pi.$$

39 *Solution*

Using the substitution $x = 5\sin^2\theta + \cos^2\theta = 4\sin^2\theta + 1$, we get $dx = 8\sin\theta\cos\theta d\theta$,

$x = 2 \Rightarrow \theta = \frac{\pi}{6}, x = 3 \Rightarrow \theta = \frac{\pi}{4}$, since $x = 4\sin^2\theta + 1$.

Hence

$$\int_2^3 \frac{1}{2\sqrt{(x-1)(5-x)}}dx = \int_{\pi/6}^{\pi/4} \frac{8\sin\theta\cos\theta}{2\sqrt{(5\sin^2\theta+\cos^2\theta-1)(5-5\sin^2\theta-\cos^2\theta)}}d\theta$$

$$= 4\int_{\pi/6}^{\pi/4} \frac{\sin\theta\cos\theta}{2\sin\theta 2\cos\theta}d\theta = \int_{\pi/6}^{\pi/4} d\theta = \frac{\pi}{12}.$$

40 *Solution*

Let $n = 0$, then integration by parts leads to

$\int \ln x dx = x\ln x - \int x(\ln x)' dx = x\ln x - \int dx = x(\ln x - 1) + c$.

Let $n = 1$, then $\int \frac{\ln x}{x}dx = \frac{1}{2}\ln^2 x + c$.

Let $n \neq 0$ or 1, then integration by parts leads to

$$\int \frac{\ln x}{x^n} dx = -\frac{\ln x}{x^{n-1}} \cdot \frac{1}{n-1} + \frac{1}{n-1} \int \frac{1}{x^{n-1}} (\ln x)' dx = -\frac{\ln x}{x^{n-1}} \cdot \frac{1}{n-1} + \frac{1}{n-1} \int \frac{1}{x^n} dx$$
$$= -\frac{\ln x}{x^{n-1}} \cdot \frac{1}{n-1} - \frac{1}{(n-1)^2} \cdot \frac{1}{x^{n-1}} + c = \frac{1}{(n-1)x^{n-1}} \left\{ \frac{1}{1-n} - \ln x \right\} + c.$$

41 *Solution*

(a) Using the substitution $u = x^2 + 1, du = 2xdx$, we get

$$\int \frac{x^3}{(x^2+1)^3} dx = \int \frac{x^2+1-1}{(x^2+1)^3} x dx = \frac{1}{2} \int \frac{u}{u^3} du - \frac{1}{2} \int \frac{1}{u^3} du = -\frac{1}{2u} + \frac{1}{4u^2} + c$$
$$= -\frac{1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + c.$$

(b) Using the substitution $x = \tan\theta$, we have

$$\int \frac{x^3}{(x^2+1)^3} dx = \int \frac{\tan^3\theta}{(\tan^2\theta+1)^3} \sec^2\theta d\theta = \int \frac{\tan^3\theta}{\sec^6\theta} \sec^2\theta d\theta = \int \tan^3\theta \cos^4\theta d\theta$$
$$= \int \sin^3\theta \cos\theta d\theta = \frac{1}{4}\sin^4\theta + c = \frac{1}{4}\left(\frac{\tan^2\theta}{1+\tan^2\theta}\right)^2 + c = \frac{1}{4}\left(\frac{x^2}{1+x^2}\right)^2 + c = \frac{1}{4}\left(\frac{x^2+1-1}{1+x^2}\right)^2 + c$$
$$= \frac{1}{4}\left(1 - \frac{1}{1+x^2}\right)^2 + c = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{1+x^2} + \frac{1}{4(1+x^2)^2} + c.$$

This is in complete agreement with result derived in (a).

42 *Solution*

(a) Make the substitution $x = a\sec\theta$. Then $\sqrt{x^2 - a^2} = a\tan\theta$, and $dx = a\dfrac{\sin\theta}{\cos^2\theta} d\theta$.

Hence

$$\int \sqrt{x^2 - a^2} dx = a^2 \int \tan\theta \frac{\sin\theta}{\cos^2\theta} d\theta = a^2 \int \frac{1-\cos^2\theta}{\cos^3\theta} d\theta = a^2 \int \frac{1}{\cos^3\theta} d\theta - a^2 \int \frac{1}{\cos\theta} d\theta$$
$$= a^2 \int \sec^3\theta d\theta - a^2 \int \sec\theta d\theta.$$

Using the recurrence formula $\int \sec^{2n+1} x dx = \dfrac{1}{2n} \cdot \dfrac{\sin x}{\cos^{2n} x} + \left(1 - \dfrac{1}{2n}\right) \int \sec^{2n-1} x dx$, we have

$$\int \sec^3\theta d\theta = \frac{1}{2} \frac{\sin\theta}{\cos^2\theta} + \frac{1}{2} \int \sec\theta d\theta = \frac{1}{2} \frac{\sin\theta}{\cos^2\theta} + \frac{1}{2} \ln|\sec\theta + \tan\theta| + c,$$

since

$$\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + c.$$

This way, we get

$$\int\sqrt{x^2-a^2}\,dx=\frac{a^2}{2}\cdot\frac{\sin\theta}{\cos^2\theta}-\frac{a^2}{2}\ln\left|\sec\theta+\tan\theta\right|+c=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c.$$

(b) Integration by parts yields

$$I=\int\sqrt{x^2-a^2}\,dx=x\sqrt{x^2-a^2}-\int x\left(\sqrt{x^2-a^2}\right)'dx=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,dx-a^2\int\frac{1}{\sqrt{x^2-a^2}}dx.$$

Hence $I=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{a^2}{2}\int\frac{1}{\sqrt{x^2-a^2}}dx=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c.$

43 *Solution*

Let $I_n=\int_0^{\pi/2}\frac{\sin(2n+1)\theta}{\sin\theta}d\theta,\quad n\geq 0$. Since $\sin(2n+1)\theta-\sin(2n-1)\theta=2\cos 2n\theta\sin\theta$,

we get for $n\geq 1$

$$I_n-I_{n-1}=2\int_0^{\pi/2}\frac{\cos 2n\theta\sin\theta}{\sin\theta}d\theta=2\int_0^{\pi/2}\cos 2n\theta d\theta=\frac{1}{n}\left[\sin 2n\theta\right]_0^{\pi/2}=0.$$

Hence $I_n-I_{n-1}=0$, and $I_n=I_0=\int_0^{\pi/2}\frac{\sin\theta}{\sin\theta}d\theta=\frac{\pi}{2}\quad$ for $n\geq 1$.

44 *Solution*

Let $I_n=\int_0^{\pi/2}\frac{\cos(2n+1)\theta}{\cos\theta}d\theta,\ n\geq 0$. Since $\cos(2n+1)\theta+\cos(2n-1)\theta=2\cos 2n\theta\cos\theta$,

we get for $n\geq 1$

$$I_n+I_{n-1}=2\int_0^{\pi/2}\frac{\cos 2n\theta\cos\theta}{\cos\theta}d\theta=2\int_0^{\pi/2}\cos 2n\theta d\theta=\frac{1}{n}\left[\sin 2n\theta\right]_0^{\pi/2}=0.$$

Hence $I_n=-I_{n-1}$, and $I_n=(-1)^nI_0=(-1)^n\int_0^{\pi/2}d\theta=(-1)^n\frac{\pi}{2}$.

45 *Solution*

Integration by parts yields for $n\geq 1$

$$I_n=\int_0^a (a^2-x^2)^n dx=\left[x(a^2-x^2)^n\right]_0^a-\int_0^a x\left((a^2-x^2)^n\right)' dx=2n\int_0^a x^2(a^2-x^2)^{n-1}dx$$

$$=2n\int_0^a (x^2-a^2+a^2)(a^2-x^2)^{n-1}dx=-2n\int_0^a (a^2-x^2)^n dx+2na^2\int_0^a (a^2-x^2)^{n-1}dx$$

$$=-2nI_n+2na^2I_{n-1}.$$

Hence $I_n=\dfrac{2n}{1+2n}a^2I_{n-1},\ n\geq 1$.

46 *Solution*

Let for $n\geq 0$

$$I_n=\int_0^{\pi/2}\sin^n x\cos^2 x dx.$$

Then integration by parts yields for $n\geq 2$

$$I_n=\int_0^{\pi/2}\sin^n x\cos^2 x dx=\frac{1}{n+1}\left[\cos x\sin^{n+1}x\right]_0^{\pi/2}-\frac{1}{n+1}\int_0^{\pi/2}\sin^{n+1}x d\cos x=\frac{1}{n+1}\int_0^{\pi/2}\sin^{n+2}x dx.$$

It is clear that

$$I_{n-2}=\frac{1}{n-1}\int_0^{\pi/2}\sin^n x dx \text{ for } n\geq 2.$$

On the other hand $I_n=\dfrac{1}{n+1}\displaystyle\int_0^{\pi/2}\sin^{n+2}x dx=\dfrac{1}{n+1}\int_0^{\pi/2}\sin^n x(1-\cos^2 x)dx$

$$=\frac{1}{n+1}\int_0^{\pi/2}\sin^n x dx-\frac{1}{n+1}\int_0^{\pi/2}\sin^n x\cos^2 x dx=\frac{n-1}{n+1}I_{n-2}-\frac{1}{n+1}I_n.$$

Hence

$$I_n\frac{n+2}{n+1}=\frac{n-1}{n+1}I_{n-2},\quad I_n=\frac{n-1}{n+2}I_{n-2},\ n\geq 2.$$

Furthermore $I_4=\displaystyle\int_0^{\pi/2}\sin^4 x\cos^2 x dx=\frac{3}{6}I_2=\frac{3}{6}\cdot\frac{1}{4}I_0=\frac{1}{8}\int_0^{\pi/2}\cos^2 x dx=\frac{1}{16}\left(\int_0^{\pi/2}1dx+\int_0^{\pi/2}\cos 2x dx\right)$

$$=\frac{1}{16}\left(\frac{\pi}{2}+\frac{1}{2}\left[\sin 2x\right]_0^{\pi/2}\right)=\frac{\pi}{32}.$$

47 *Solution*

Repeated application of integration by parts can be used to reduce the power of x in the integrand stepwise in the following way:

$$I_n = \int_0^{\pi/2} x^n \sin x dx = -\Big[\cos x \cdot x^n\Big]_0^{\pi/2} + n\int_0^{\pi/2} \cos x \cdot x^{n-1} dx$$

$$= n\Big[\sin x \cdot x^{n-1}\Big]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} \sin x \cdot x^{n-2} dx = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

for $n \geq 2$. Hence

$$I_4 = \int_0^{\pi/2} x^4 \sin x dx = 4\left(\frac{\pi}{2}\right)^3 - 12I_2 = 4\left(\frac{\pi}{2}\right)^3 - 12\left(2\frac{\pi}{2} - 2I_0\right) = \frac{\pi^3}{2} - 12\pi + 24\int_0^{\pi/2} \sin x dx$$

$$= \frac{\pi^3}{2} - 12\pi + 24.$$

48 *Solution*

Repeated application of integration by parts to reduce the power of $\cos^n x$ in the integrand stepwise leads to the following:

$$I_n = \int_0^{\pi/2} \cos^n x dx = \Big[\sin x \cdot \cos^{n-1} x\Big]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot \sin^2 x dx$$

$$= (n-1)\int_0^{\pi/2} \cos^{n-2} x \cdot (1-\cos^2 x) dx = (n-1)\int_0^{\pi/2} \cos^{n-2} x dx - (n-1)\int_0^{\pi/2} \cos^n x dx$$

$$= (n-1)I_{n-2} - (n-1)I_n.$$

Hence for $n \geq 2$

$$n \cdot I_n = (n-1)I_{n-2} \Rightarrow I_n = \frac{n-1}{n} I_{n-2}.$$

Furthermore, we get

$$I_{10} = \frac{9}{10} I_8 = \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} I_0 = \frac{63}{512}\pi.$$

Using the substitution $2\theta = u$, we get

$$\int_{-\pi/4}^{\pi/4} \cos^{10} 2\theta d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^{10} u du = \int_0^{\pi/2} \cos^{10} u du = I_{10} = \frac{63}{512}\pi.$$

49 *Solution*

Make the substitution $u = \frac{\pi}{2} - x,\ du = -dx,\ x = 0 \Rightarrow u = \frac{\pi}{2},\ x = \pi \Rightarrow u = -\frac{\pi}{2}$. Then

$$\int_0^{\pi}\left(x - \frac{\pi}{2}\right)^6 \cos^3 x dx = -\int_{\pi/2}^{-\pi/2} (-u)^6 \cos^3\left(\frac{\pi}{2} - u\right) du = \int_{-\pi/2}^{\pi/2} u^6 \sin^3 u du = \int_{-\pi/2}^{\pi/2} x^6 \sin^3 x dx = 0,$$

since $x^6 \sin^3 x$ is odd.

50 *Solution*

Let us show that $\int_0^a f(x)dx = \int_0^{a/2} \{f(x) + f(a-x)\}dx$.

It is easily seen that $\int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_{a/2}^a f(x)dx$. Make the substitution $x = a - u$ in the second integral. Hence

$$\int_0^a f(x)dx = \int_0^{a/2} f(x)dx - \int_{a/2}^0 f(a-u)du = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-u)du = \int_0^{a/2} \{f(x) + f(a-x)\}dx.$$

Using this relation, we get

$$\int_0^{\pi} x\sin^6 xdx = \int_0^{\pi/2} \left\{x\sin^6 x + (\pi - x)\sin^6(\pi - x)\right\}dx = \int_0^{\pi/2} \left\{x\sin^6 x + \pi\sin^6 x - x\sin^6 x\right\}dx$$

$$= \pi \int_0^{\pi/2} \sin^6 xdx = \frac{5\pi^2}{32},$$

since

$$I_n = \int_0^{\pi/2} \sin^n xdx = -\left[\sin^{n-1} x\cos x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x\cos^2 xdx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x(1 - \sin^2 x)dx = (n-1)(I_{n-2} - I_n),$$

$I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \ge 2$ and $I_6 = \dfrac{5}{6} I_4 = \dfrac{5}{6}\cdot\dfrac{3}{4}\cdot\dfrac{1}{2} I_0 = \dfrac{15}{48}\cdot\dfrac{\pi}{2} = \dfrac{5}{32}\pi.$