7SD Solutions Series

Worked Solutions to Popular Mathematics Texts

Suggested Worked Solutions to

"4 Unit Mathematics"

(Text book for the NSW HSC by D. Arnold and G. Arnold)

Chapter 2
Complex Numbers

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Solutions are to "4 Unit Mathematics"

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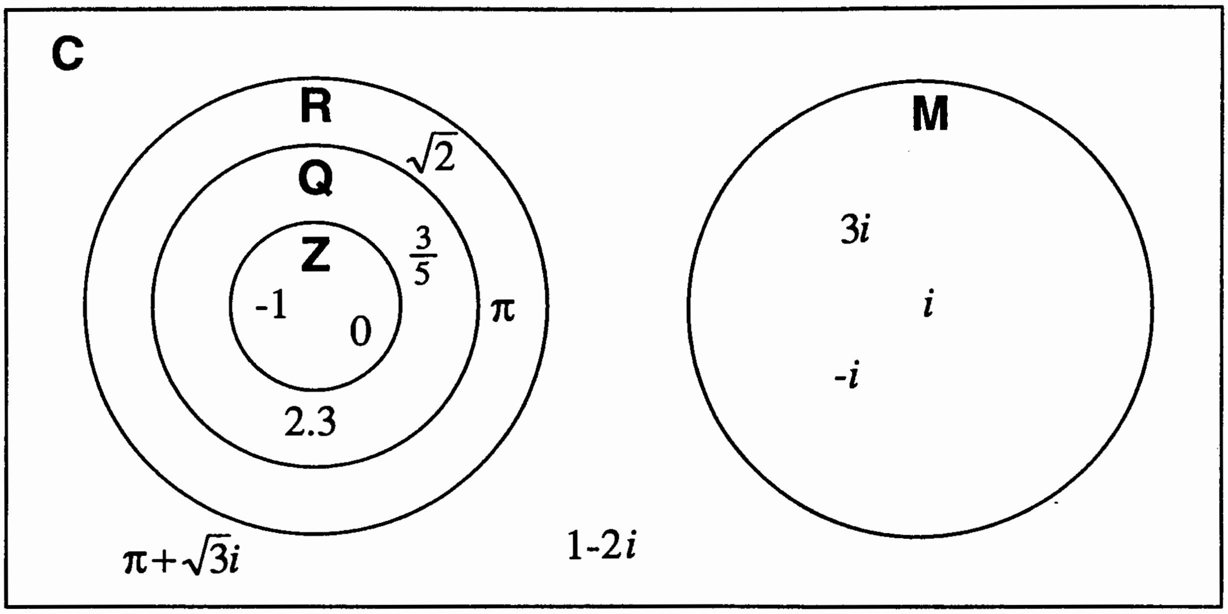
Exercise 2.1

1 *Solution*



2 *Solution*

(a) $z_1 + z_2 = 3 + i$

(b) $z_1 - z_2 = 1 - 7i$

(c) $z_1 z_2 = 2 - 12i^2 - 3i + 8i = 14 + 5i$

(d) $z_1^2 = 4 - 12i + 9i^2 = -5 - 12i$

(e) $\dfrac{1}{z_2} = \dfrac{1}{1+4i} = \dfrac{1-4i}{(1+4i)(1-4i)} = \dfrac{1-4i}{1+16} = \dfrac{1}{17} - \dfrac{4}{17}i$

(f) $\dfrac{z_2}{z_1} = \dfrac{1+4i}{2-3i} = \dfrac{(1+4i)(2+3i)}{(2-3i)(2+3i)} = \dfrac{(2-12)+(8+3)i}{4+9} = -\dfrac{10}{13} + \dfrac{11}{13}i$

(g) $z_1^2 - z_2^2 = (z_1 - z_2)(z_1 + z_2) = (1-7i)(3+i) = (3+7) + (-21+1)i = 10 - 20i$

(h) $z_1^3 - z_2^3 = (z_1 - z_2)\left(z_1^2 + z_1 z_2 + z_2^2\right) =$

$$= (1-7i)\left((-5-12i) + (14+5i) + \left(1 + 8i + 16i^2\right)\right) =$$

$$= (1-7i)(-6+i) = (-6+7) + (42+1)i = 1 + 43i$$

3 *Solution*

(a) $\bar{z} = -3 - 2i \quad z\bar{z} = (-3+2i)(-3-2i) = 9 + 4 = 13 \in \mathbf{R}$

(b) $\dfrac{1}{z} = \dfrac{1}{-3+2i} = \dfrac{-3-2i}{13} = -\dfrac{3}{13} - \dfrac{2}{13}i$

4 *Solution*

Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2,\ x_1, y_1, x_2, y_2 \in \mathbf{R}$. Then

(a) $\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = \bar{z}_1 + \bar{z}_2$

(b) $\overline{z_1 - z_2} = \overline{(x_1 - x_2) + i(y_1 - y_2)} = (x_1 - x_2) - i(y_1 - y_2) = \bar{z}_1 - \bar{z}_2$

(c) $\overline{z_1 z_2} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} =$

$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) =$

$= \bar{z}_1\bar{z}_2$

(e) $$\overline{z_1 \div z_2} = \overline{\left\{\frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}\right\}} = \overline{\left(\frac{x_1}{x_2^2 + y_2^2} + i\frac{y_1}{x_2^2 + y_2^2}\right)\left(\frac{x_2}{x_2^2 + y_2^2} - i\frac{y_2}{x_2^2 + y_2^2}\right)} =$$

$$= \left(\frac{x_1}{x_2^2 + y_2^2} - i\frac{y_1}{x_2^2 + y_2^2}\right)\left(\frac{x_2}{x_2^2 + y_2^2} + i\frac{y_2}{x_2^2 + y_2^2}\right) =$$

$$= \frac{(x_1 - iy_1)(x_2 + iy_2)}{x_2^2 + y_2^2} = \frac{x_1 - iy_1}{x_2 - iy_2} = \overline{z_1} \div \overline{z_2}$$

(d) Identity $\overline{\left(\frac{1}{z}\right)} = \frac{1}{(\bar{z})}$ follows from (e) with $z_1 = 1$ and $z_2 = z$

Identity $\overline{5z} = 5\bar{z}$ follows from (c) with $z_1 = 5$ and $z_2 = z$

5 *Solution*

(a) Using the results in question 4 gives

$$a(\bar{\alpha})^2 + b\bar{\alpha} + c = \overline{(a\alpha^2)} + \overline{b\alpha} + \bar{c} = \overline{a\alpha^2 + b\alpha + c} = \bar{0} = 0.$$

(b) If α is a non-real number, then $\operatorname{Im}\alpha \neq 0$. Hence $\bar{\alpha} \neq \alpha$, since $\operatorname{Im}(\bar{\alpha}) = -\operatorname{Im}\alpha$.

Thus if α is a non-real root of $ax^2 + bx^2 + c = 0$, where a, b, c are real, then $\bar{\alpha}$ is the other root of this quadratic equation (see (a)).

6 *Solution*

(a) $\operatorname{Im} z = 2 \Rightarrow z = x + 2i$ and $z^2 = (x^2 - 4) + i(4x),\ x \in \mathbf{R}$

z^2 real $\Rightarrow 4x = 0 \Rightarrow x = 0,$

$\therefore\ z = 2i.$

(b) $\operatorname{Re} z = 2\operatorname{Im} z \Rightarrow z = 2y + iy$ and $z^2 - 4i = (4y^2 - y^2) + i(4y^2 - 4),\ y \in \mathbf{R}$

$z^2 - 4i$ real $\Rightarrow 4y^2 - 4 = 0 \Rightarrow y = \pm 1,$

$\therefore\ z = 2 + i$ or $z = -2 - i.$

7 *Solution*

$\dfrac{z}{z-i}$ is real $\Rightarrow$ $\dfrac{z-i+i}{z-i}=1+\dfrac{i}{z-i}=1+\dfrac{i\cdot i}{i(z-i)}=1-\dfrac{1}{iz+1}$ is real.

$\therefore$ $\dfrac{1}{iz+1}$ is real $\Rightarrow$ $\dfrac{-i\bar{z}+1}{(iz+1)(-i\bar{z}+1)}$ is real. Hence $i\bar{z}$ is real $\Rightarrow$ $i(i\bar{z})$ is imaginary.

Thus $\bar{z}$ is imaginary $\Rightarrow$ z is imaginary.

8 *Solution*

(a) $-25=25i^2$, $\therefore$ -25 has square roots $5i$ and $-5i$.

(b) Let $(a+ib)^2=-6i$, a, $b\in\mathbf{R}$. Then $\left(a^2-b^2\right)+i(2ab)=-6i$. Equating real and imaginary parts, $a^2-b^2=0$ and $2ab=-6$.

$a^2-\dfrac{9}{a^2}=0\Rightarrow a^4-9=0$

$\left(a^2-3\right)\left(a^2+3\right)=0$, a real $\Rightarrow$ $a=\sqrt{3}$, $b=-\sqrt{3}$ or $a=-\sqrt{3}$, $b=\sqrt{3}$. Hence $-6i$ has square roots $\sqrt{3}-i\sqrt{3}$, $-\sqrt{3}+i\sqrt{3}$.

(c) Let $(a+ib)^2=i$, a, $b\in\mathbf{R}$. Then $\left(a^2-b^2\right)+i(2ab)=i$. Equating real and imaginary parts, $a^2-b^2=0$ and $2ab=1$.

$a^2-\dfrac{1}{4a^2}=0\Rightarrow 4a^4-1=0$

$\left(2a^2-1\right)\left(2a^2+1\right)=0$, a real $\Rightarrow$ $a=\dfrac{1}{\sqrt{2}}$, $b=\dfrac{1}{\sqrt{2}}$ or $a=-\dfrac{1}{\sqrt{2}}$, $b=-\dfrac{1}{\sqrt{2}}$. Hence i has square roots $\dfrac{1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}-i\dfrac{1}{\sqrt{2}}$.

(d) Let $(a+ib)^2=-4+3i$, a, $b\in\mathbf{R}$. Then $\left(a^2-b^2\right)+i(2ab)=-4+3i$. Equating real and imaginary parts, $a^2-b^2=-4$ and $2ab=3$.

$a^2-\dfrac{9}{4a^2}=-4\Rightarrow 4a^4+16a^2-9=0$

$\left(2a^2-1\right)\left(2a^2+9\right)=0$, a real $\Rightarrow$ $a=\dfrac{1}{\sqrt{2}}$, $b=\dfrac{3}{\sqrt{2}}$ or $a=-\dfrac{1}{\sqrt{2}}$, $b=-\dfrac{3}{\sqrt{2}}$. Hence $-4+3i$ has square roots $\dfrac{1}{\sqrt{2}}+i\dfrac{3}{\sqrt{2}}$, $-\dfrac{1}{\sqrt{2}}-i\dfrac{3}{\sqrt{2}}$.

(e) Let $(a+ib)^2=-5-12i$, a, $b\in\mathbf{R}$. Then $\left(a^2-b^2\right)+i(2ab)=-5-12i$. Equating real and imaginary parts, $a^2-b^2=-5$ and $2ab=-12$.

$a^2-\dfrac{36}{a^2}=-5\Rightarrow a^4+5a^2-36=0$

$\left(a^2-4\right)\left(a^2+9\right)=0$, a real $\Rightarrow$ $a=2$, $b=-3$ or $a=-2$, $b=3$. Hence $-5-12i$ has square roots $2-3i$ $-2+3i$.

9 *Solution*

(a) $\Delta = -3 = 3i^2$; $\therefore$ $x = \dfrac{-1 \pm i\sqrt{3}}{2} = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

(b) $\Delta = -8 = 8i^2$, $\therefore$ $x = \dfrac{4 \pm i\sqrt{8}}{4} = 1 \pm i\dfrac{1}{\sqrt{2}}$

(c) Find Δ: $16(1+2i)^2 + 16(3-4i) = 0$. Hence $4x^2 - 4(1+2i)x - (3-4i) = 0$ has two equal solutions $x = \dfrac{1}{2} + i$.

(d) Find Δ: $4(1+i)^2 - 40i = -32i$.

Find square roots of Δ: Let $(a+ib)^2 = -32i$, $a, b \in \mathbf{R}$. Then $(a^2 - b^2) + i(2ab) = -32i$. Equating real and imaginary parts, $a^2 - b^2 = 0$ and $ab = -16$. $a^2 - \dfrac{16^2}{a^2} = 0 \Rightarrow a^4 - 16^2 = 0$

$(a^2 - 16)(a^2 + 16) = 0$, a real $\Rightarrow a = 4, b = -4$ or $a = -4$, $b = 4$. Hence Δ has square roots $\pm(4-4i)$.

Use the quadratic formula: $ix^2 - 2(i+1)x + 10 = 0$ has solutions $x = \dfrac{2(1+i) \pm 4(1-i)}{2i}$, $\therefore$ $x = -1-3i$ or $x = 3+i$.

10 *Solution*

(a) b and c are real, $\therefore$ $3+2i$ is the other root of $x^2 + bx + c = 0$. Hence $c = (3-2i)(3+2i)$ and $-b = (3-2i) + (3+2i)$. Thus $c = 9 + 4 = 13$ and $b = -6$.

(b) $\operatorname{Im} \alpha = 2 \Rightarrow \alpha = x + 2i$, $x \in \mathbf{R}$. k real $\Rightarrow \overline{\alpha} = x - 2i$ is the other root of $x^2 + 6x + k = 0$. Hence $k = (x+2i)(x-2i)$ and $-6 = (x+2i) + (x-2i)$.

$\therefore$ $k = x^2 + 4$ and $-6 = 2x$. Thus $x = -3$ and $k = 13$. Hence both roots of the equation are $-3 \pm 2i$.

(c) Let z be the other root of $x^2 - (3+i)x + k = 0$. Then $3 + i = (1-2i) + z$.

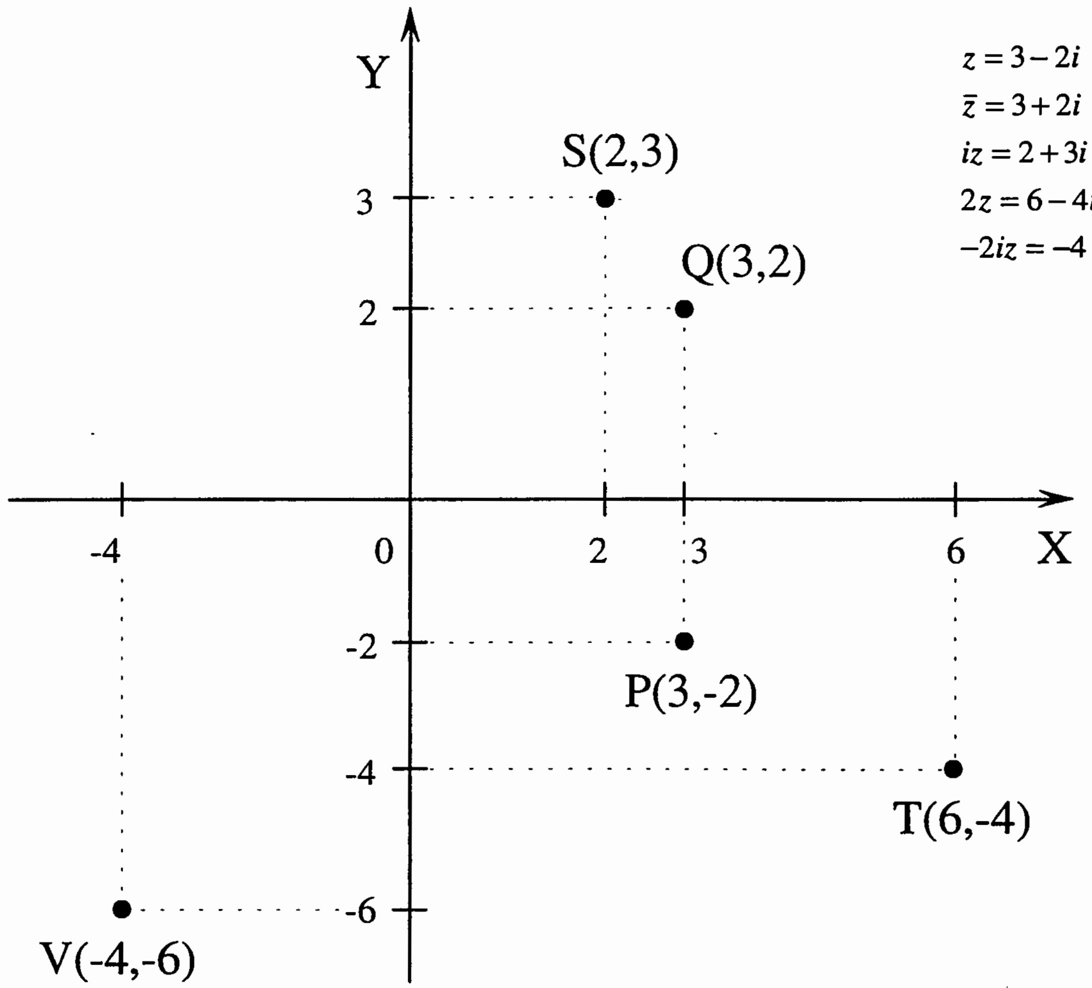
$\therefore$ $z = (3+i) - (1-2i) = 2 + 3i$. Hence

$k = (1-2i)z = (1-2i)(2+3i) = (2+6) + i(-4+3) =$

$= 8 - i$.

Exercise 2.2

1 ***Solution***



$z = 3 - 2i$

$\bar{z} = 3 + 2i$

$iz = 2 + 3i$

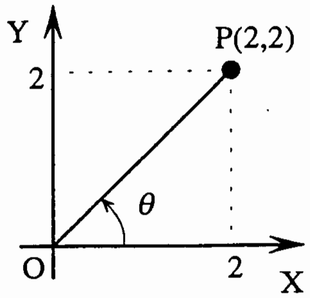
$2z = 6 - 4i$

$-2iz = -4 - 6i$

2 ***Solution***

In each case $P(a,b)$ represents the complex number $z = a + ib$ and θ is the principal argument of z

(a)

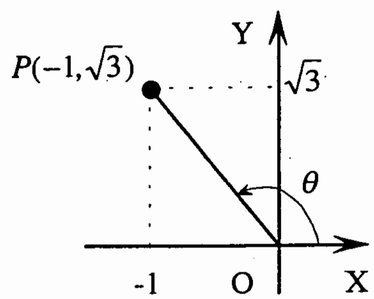


$z = 2 + 2i$

$|z| = \sqrt{4+4} = 2\sqrt{2}$

$\arg\ z = \dfrac{\pi}{4}$

(b)

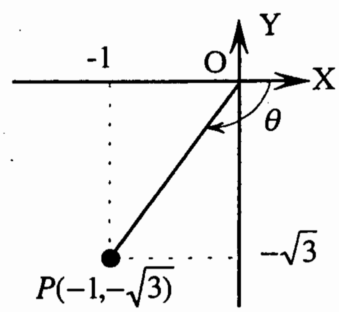


$z = -1 + \sqrt{3}i$

$|z| = \sqrt{1+3} = 2$

$\theta = \pi - \dfrac{\pi}{3} \Rightarrow \arg\ z = \dfrac{2\pi}{3}$

(c)

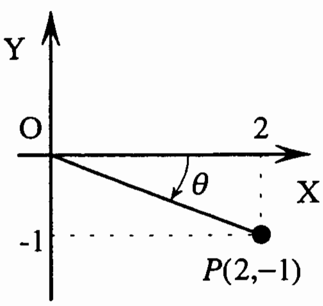


$z = -1 - \sqrt{3}i$

$|z| = \sqrt{1+3} = 2$

$\theta = -\pi + \dfrac{\pi}{3} \Rightarrow \arg\ z = -\dfrac{2\pi}{3}$

(d)

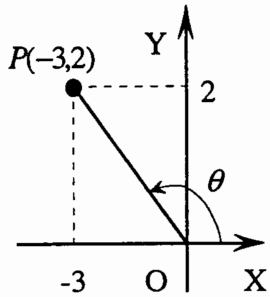


$z = 2 - i$

$|z| = \sqrt{4+1} = \sqrt{5}$

$\arg\ z = -\tan^{-1}(1/2)$

(e)

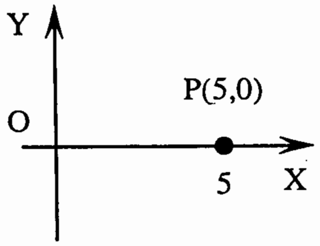


$z=-3+2i$

$|z|=\sqrt{9+4}=\sqrt{13}$

$\theta=\pi-\tan^{-1}(2/3)\Rightarrow \arg z=\pi-\tan^{-1}(2/3)$

(f)

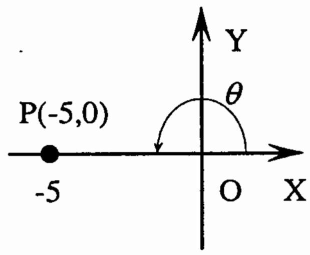


$z=5$

$|z|=5$

$\theta=0\Rightarrow \arg\ z=0$

(g)

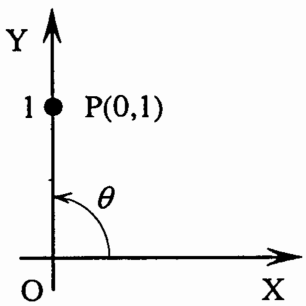


$z=-5$

$|z|=5$

$\arg\ z=\pi$

(h)

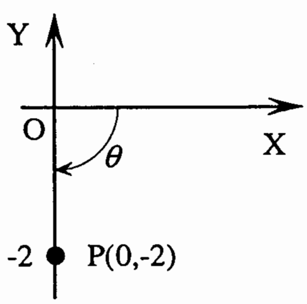


$z=i$

$|z|=1$

$\arg\ z=\dfrac{\pi}{2}$

(i)

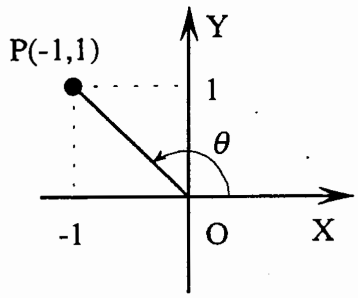


$z=-2i$

$|z|=2$

$\arg\ z=-\dfrac{\pi}{2}$

(j)

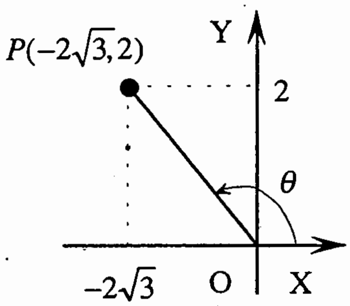


$z = i(i+1) = -1 + i$

$|z| = \sqrt{1+1} = \sqrt{2}$

$\theta = \pi - \frac{\pi}{4} \Rightarrow \arg\ z = \dfrac{3\pi}{4}$

3 *Solution*



Let $z = -2\sqrt{3} + 2i$

$|z| = \sqrt{12+4} = 4$

$\theta = \pi - \dfrac{\pi}{6} \Rightarrow \arg\ z = \dfrac{5\pi}{6}$

$z = 4\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right) = 4\text{cis}\dfrac{5\pi}{6}$

4 *Solution*

Let $r_1 = |z_1|$, $r_2 = |z_2|$ and $\theta_1 = \arg\ z_1$, $\theta_2 = \arg\ z_2$. Then

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $\quad \bar{z}_1 = r_1(\cos(-\theta_1) + i\sin(-\theta_1))$,

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $\quad \bar{z}_2 = r_2(\cos(-\theta_2) + i\sin(-\theta_2))$.

(a) $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \Rightarrow$

$\Rightarrow \overline{z_1 z_2} = r_1 r_2(\cos(-(\theta_1 + \theta_2)) + i\sin(-(\theta_1 + \theta_2)))$.

But $\overline{z_1} \cdot \overline{z_2} = r_1 r_2(\cos((-\theta_1) + (-\theta_2)) + i\sin((-\theta_1) + (-\theta_2)))$. Therefore, $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(b) Let $r = |z|$ and $\arg z = 0$. Then $z = r(\cos\theta + i\sin\theta)$, $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$

and $\dfrac{1}{z} = \dfrac{1}{r}(\cos(-\theta) + i\sin(-\theta))$. Thus $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{r}(\cos\theta + i\sin\theta)$ and

$\dfrac{1}{(\bar{z})} = \dfrac{1}{r}(\cos\theta + i\sin\theta)$. Hence $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{(\bar{z})}$.

(c)

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right)\Rightarrow\overline{\left(\frac{z_1}{z_2}\right)}=\frac{r_1}{r_2}\left(\cos(-(\theta_1-\theta_2))+i\sin(-(\theta_1-\theta_2))\right).$$

But $\frac{(\bar{z}_1)}{(\bar{z}_2)}=\frac{r_1}{r_2}\left(\cos(-\theta_1+\theta_2)+i\sin(-\theta_1+\theta_2)\right)$. Therefore, $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{(\overline{z_1})}{(\overline{z_2})}$.

5 *Solution*

Define the statement $S(n)$: $|z^n|=|z|^n$ and $\arg(z^n)=n\arg z$, $n=1,2,K$ Clearly $S(1)$ is true. If $S(k)$ is true, then $|z^k|=|z|^k$ and $\arg(z^k)=k\arg z$. Consider $S(k+1)$.

$|z^{k+1}|=|z^k\cdot z|=|z^k|\cdot|z|=|z|^k\cdot|z|$, if $S(k)$ is true.

$\therefore |z^{k+1}|=|z|^{k+1}$, if $S(k)$ is true.

$\arg(z^{k+1})=\arg(z^k\cdot z)=\arg(z^k)+\arg z=k\arg z+\arg z$, if $S(k)$ is true.

$\therefore \arg(z^{k+1})=(k+1)\arg z$, if $S(k)$ is true.

Hence for all positive integers k, $S(k)$ true implies $S(k+1)$ true. But $S(1)$ is true, therefore by induction, $S(n)$ is true for all positive integers n.

$\therefore |z^n|=|z|^n$ and $\arg(z^n)=n\arg z$ for all positive integers n.

6 *Solution*

(a) $|z_1|=4\Rightarrow|z_1^3|=4^3=64$.

$\arg z_1=\frac{\pi}{3}\Rightarrow\arg(z_1^3)=3\cdot\frac{\pi}{3}=\pi$.

$\therefore z_1^3$ has modulus 64 and principal argument π.

(b) $|z_2|=2\Rightarrow\left|\frac{1}{z_2}\right|=\frac{1}{2}$.

$$\arg z_2 = \frac{\pi}{6} \Rightarrow \arg\left(\frac{1}{z_2}\right) = -\frac{\pi}{6}.$$

$\therefore \dfrac{1}{z_2}$ has modulus $\frac{1}{2}$ and principal argument $-\dfrac{\pi}{6}$.

(c) $$\frac{z_1^3}{z_2} = z_1^3 \cdot \left(\frac{1}{z_2}\right) \Rightarrow \begin{cases} \left|\dfrac{z_1^3}{z_2}\right| = \left|z_1^3\right| \cdot \left|\dfrac{1}{z_2}\right| = 64 \cdot \dfrac{1}{2} = 32 \\ \arg\left(\dfrac{z_1^3}{z_2}\right) = \arg\left(z_1^3\right) + \arg\left(\dfrac{1}{z_2}\right) = \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}. \end{cases}$$

$\therefore \dfrac{z_1^3}{z_2}$ has modulus 32 and principal argument $\dfrac{5\pi}{6}$.

7 *Solution*

Let $z_1 = -\sqrt{3} + i$ and $z_2 = 4 + 4i$. Then

$$z_1 = 2\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \Rightarrow |z_1| = 2,\ \arg z_1 = \frac{5\pi}{6},$$

$$z_2 = 4\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow |z_2| = 4\sqrt{2},\ \arg z_2 = \frac{\pi}{4}.$$

(a) $\left(-\sqrt{3} + i\right)(4 + 4i) = z_1 z_2$. But $|z_1 z_2| = |z_1| \cdot |z_2| = 8\sqrt{2}$ and

$\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \dfrac{5\pi}{6} + \dfrac{\pi}{4} = \dfrac{13\pi}{12}$. Since $\dfrac{13\pi}{12} > \pi$, the principal argument of

$z_1 z_2$ is $\dfrac{13\pi}{12} - 2\pi = -\dfrac{11\pi}{12}$. Hence

$$\left(-\sqrt{3} + i\right)(4 + 4i) = 8\sqrt{2}\left[\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right] = 8\sqrt{2}\,\mathrm{cis}\left(-\frac{11\pi}{12}\right)$$

(b) $\dfrac{-\sqrt{3} + i}{4 + 4i} = \dfrac{z_1}{z_2}$. But $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} = \dfrac{1}{2\sqrt{2}}$ and

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \dfrac{5\pi}{6} - \dfrac{\pi}{4} = \dfrac{7\pi}{12}$. Hence

$$\frac{-\sqrt{3} + i}{4 + 4i} = \frac{1}{2\sqrt{2}}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) = \frac{1}{2\sqrt{2}}\mathrm{cis}\frac{7\pi}{12}.$$

8 *Solution*

$z=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\Rightarrow|z|=2,\ \arg z=\frac{\pi}{3}$. If z^n is real, then

$\arg\left(z^n\right)=k\pi$, k is integral. But $\arg\left(z^n\right)=n\arg z$. Therefore $n\cdot\frac{\pi}{3}=k\pi$,

$k=0,\pm1,\pm2,\mathrm{K}$,

$\therefore n=3k,\ k=0,\pm1,\pm2,\mathrm{K}$

Hence the smallest positive integer n such that z^n is real is 3.

$\left|z^3\right|=2^3=8$ and $\arg\left(z^3\right)=\pi$,

$\therefore z^3=-8$.

If z^n is imaginary, then $\arg\left(z^n\right)=\frac{\pi}{2}+k\pi$, k is integral. But $\arg\left(z^n\right)=n\arg z$.

Therefore $n\cdot\frac{\pi}{3}=\frac{\pi}{2}+k\pi,\ k=0,\pm1,\pm2,\mathrm{K}$,

$\therefore n=\frac{3}{2}+3k,\ k=0,\pm1,\pm2,\mathrm{K}$

Hence there is no integral value of n for which z^n is imaginary.

9 *Solution*

(a) $\left|z^2\right|=|z|^2=r^2$ and $\arg\left(z^2\right)=2\arg z=2\theta$

(b) $\left|\frac{1}{z}\right|=\frac{1}{|z|}=\frac{1}{r}$ and $\arg\left(\frac{1}{z}\right)=-\arg z=-\theta$

(c) $i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\Rightarrow|i|=1$ and $\arg i=\frac{\pi}{2}$. Then $|iz|=|i|\cdot|z|=1\cdot r=r$ and

$\arg(iz)=\arg(i)+\arg z=\frac{\pi}{2}+\theta$.

10 ***Solution***

(a) Let $z=1+\sqrt{3}i$. Then $z=2\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\Rightarrow|z|=2$ and

$\arg z=\frac{\pi}{3}$. Hence $\left|\frac{1}{z}\right|=\frac{1}{|z|}=\frac{1}{2}$ and $\arg\left(\frac{1}{z}\right)=-\arg z=-\frac{\pi}{3}$.

$\therefore\left(1+\sqrt{3}i\right)^{-1}=\frac{1}{2}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)=\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$.

(b) Let $z=-1+i$. Then $z=\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\Rightarrow|z|=\sqrt{2}$

and $\arg z=\frac{3\pi}{4}$. Hence $\left|z^{18}\right|=|z|^{18}=2^9=512$ and

$\arg\left(z^{18}\right)=18\arg z=18\cdot\frac{3\pi}{4}=\frac{27\pi}{2}=14\pi-\frac{\pi}{2}$.

Therefore $z^{18}=512\cdot\left(\cos\left(14\pi-\frac{\pi}{2}\right)+i\sin\left(14\pi-\frac{\pi}{2}\right)\right)=512\cdot(-i)=-512i$.

$\therefore|-1+i|=\sqrt{2}$, $\arg(-1+i)=\frac{3\pi}{4}$, $(-1+i)^{18}=-512i$.

11 ***Solution***

Let $z=\sqrt{3}+i$. Then $z=2\cdot\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\Rightarrow|z|=2$, $\arg z=\frac{\pi}{6}$. Let

$z_1=\sqrt{3}-i$. Then $z_1=\bar{z}$ and $|z_1|=|z|=2$, $\arg z_1=-\arg z=-\frac{\pi}{6}$. Hence

$\left|z^{10}\right|=|z|^{10}=2^{10}=1024$, $\left|z_1^{10}\right|=|z_1|^{10}=|z|^{10}=1024$ and

$\arg\left(z^{10}\right)=10\arg z=\frac{5\pi}{3}=2\pi-\frac{\pi}{3}$, $\arg\left(z_1^{10}\right)=10\arg z_1=-\frac{5\pi}{3}=-2\pi+\frac{\pi}{3}$. Therefore

$z^{10}+z_1^{10}=1024\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)+1024\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\cdot1024\cdot\cos\frac{\pi}{3}=1024$

.

$\therefore\sqrt{3}+i=2\operatorname{cis}\frac{\pi}{6}$, $\sqrt{3}-i=2\operatorname{cis}\left(-\frac{\pi}{6}\right)$, $\left(\sqrt{3}+i\right)^{10}+\left(\sqrt{3}-i\right)^{10}=1024$.

12 *Solution*

Let $z_1 = 7 - i$, $z_2 = 3 - 4i$, and $z = \dfrac{7-i}{3-4i}$. Then $|z_1| = \sqrt{49+1} = 5\sqrt{2}$ and

$\arg z_1 = -\tan^{-1}\left(\dfrac{1}{7}\right)$, $|z_2| = \sqrt{9+16} = 5$ and $\arg z_2 = -\tan^{-1}\left(\dfrac{4}{3}\right)$, $|z| = \dfrac{|z_1|}{|z_2|} = \sqrt{2}$ and

$\arg z = \arg z_1 - \arg z_2 = \tan^{-1}\left(\dfrac{4}{3}\right) - \tan^{-1}\left(\dfrac{1}{7}\right)$. Use a well-known formula:

$$\tan\left\{\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right)\right\} = \frac{\tan\left(\tan^{-1}\frac{4}{3}\right) - \tan\left(\tan^{-1}\frac{1}{7}\right)}{1 + \tan\left(\tan^{-1}\frac{4}{3}\right)\cdot\tan\left(\tan^{-1}\frac{1}{7}\right)} = \frac{\frac{4}{3} - \frac{1}{7}}{1 + \frac{4}{3}\cdot\frac{1}{7}} = 1.$$ Hence

$\tan\arg z = 1$. But $\dfrac{4}{3} > \dfrac{1}{7}$. Therefore $\arg z = \tan^{-1}\left(\dfrac{4}{3}\right) - \tan^{-1}\left(\dfrac{1}{7}\right) \in \left(0, \frac{\pi}{2}\right)$. Thus

principal value of argument z is $\dfrac{\pi}{4}$.

$\therefore$ Modulus of $\dfrac{7-i}{3-4i}$ is 5, $\tan\left\{\tan^{-1}\left(\dfrac{4}{3}\right) - \tan^{-1}\left(\dfrac{1}{7}\right)\right\} = 1$, principal argument of

$\dfrac{7-i}{3-4i}$ is $\dfrac{\pi}{4}$.

13 *Solution*

$$\alpha = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$

$\therefore$ $\alpha = 2\sqrt{2}\beta$, where $\beta = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}$. $z \to \alpha z$ can be expressed as

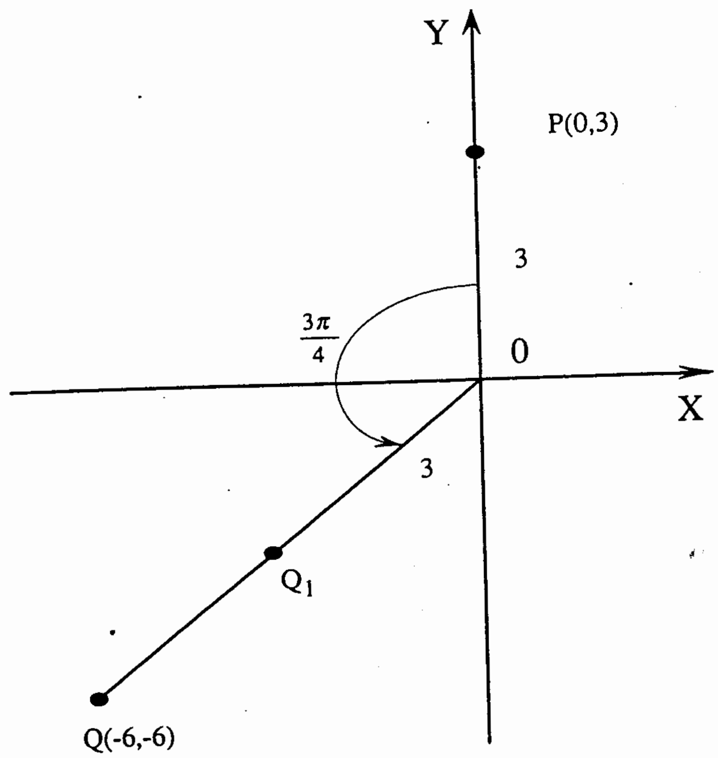
$z \to \beta z \to 2\sqrt{2}\beta z$. Let P, Q_1, Q represent z, βz, $2\sqrt{2}\beta z$ respectively. Then

$|\beta z| = |\beta|\cdot|z| = |z| \Rightarrow OQ_1 = OP$ $\arg(\beta z) = \dfrac{3\pi}{4} + \arg z \Rightarrow$ ray OQ_1 makes the angle $\dfrac{3\pi}{4}$

with ray OP. Hence $\beta \to \beta z$ is a rotation anticlockwise about P through $\dfrac{3\pi}{4}$ and

$z \to \alpha z$ is the composition of this rotation followed by an enlargement about O by

the factor $2\sqrt{2}$.



$z = 3i$, $|z| = 3$ and $\arg z = \dfrac{\pi}{2}$

$|\beta z| = 3$ and $\arg(\beta z) = \dfrac{3\pi}{4} + \dfrac{\pi}{2}$:

$|\alpha z| = 6\sqrt{2}$ and $\arg(\alpha z) = \dfrac{5\pi}{4}$

$\alpha z = -6 - 6i$

14 *Solution*

(a) Using the method of completing the square: $x^2 + px + 1 = 0 \Rightarrow \left(x + \dfrac{p^2}{2}\right) = \dfrac{p^2}{4} - 1$.

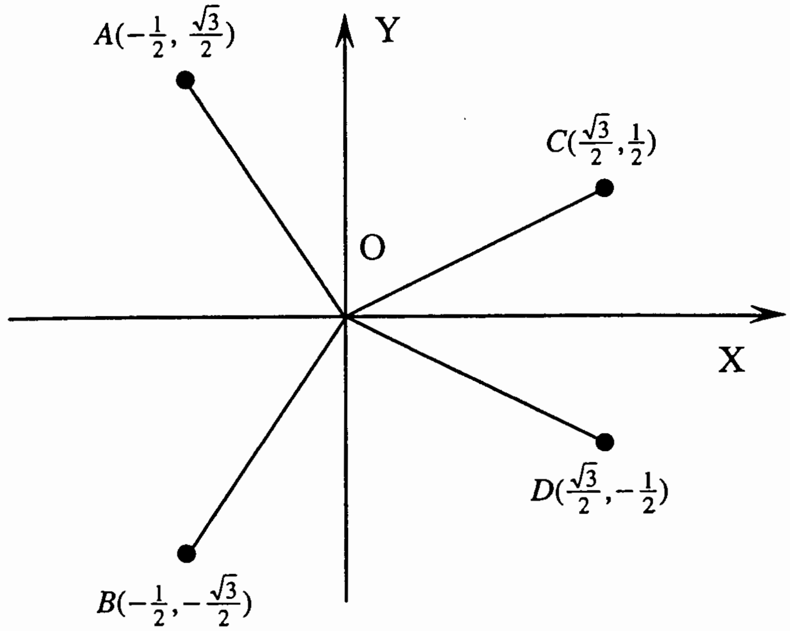
Since $-2 < p < 2$, $\dfrac{p^2}{4} - 1 < 0$. Therefore there are no real roots of the equation

$x^2 + px + 1 = 0$.

(b) Using the quadratic formula:

$$x^2 + x + 1 = 0 \Rightarrow \Delta = -3 \Rightarrow x = \frac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i,$$

$$x^2 - \sqrt{3} + 1 = 0 \Rightarrow \Delta = -1 \Rightarrow x = \frac{\sqrt{3} \pm i}{2} = \frac{\sqrt{3}}{2} \pm \frac{1}{2}i.$$



(c) Let $x_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $x_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ are the solutions of the first equation, and

$x_3 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $x_4 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ are the solutions of the second equation.

Then $\arg x_1 = \frac{2\pi}{3}$, $\arg x_2 = -\frac{2\pi}{3}$, $|x_1| = |x_2| = 1$,

$\arg x_3 = \frac{\pi}{6}$, $\arg x_4 = -\frac{\pi}{6}$, $|x_3| = |x_4| = 1$.

Hence $\angle AOB = 2\pi - (\arg x_1 - \arg x_2) = \frac{2\pi}{3}$,

$\angle COD = \arg x_3 - \arg x_4 = \frac{\pi}{3}$,

$\angle COA = \arg x_1 - \arg x_3 = \frac{\pi}{2}$.

$\angle ACB = \angle ACO + \angle BCO$.

But $\angle ACO = \frac{1}{2}(\pi - \angle AOC)$, since $AO = OC = 1$, and

$\angle BCO = \frac{1}{2}(\pi - \angle BOC)$, since $BO = OC = 1$.

Therefore $\angle ACB = \pi - \frac{1}{2}(\angle AOC + \angle BOC) = \pi - \frac{1}{2}(2\pi - \angle AOB) = \frac{1}{2}\angle AOB = \frac{\pi}{3}$.

$$\therefore \quad \angle AOB = \frac{2\pi}{3}, \quad \angle COD = \frac{\pi}{3}, \quad \angle COA = \frac{\pi}{2}, \quad \angle ACB = \frac{\pi}{3}.$$

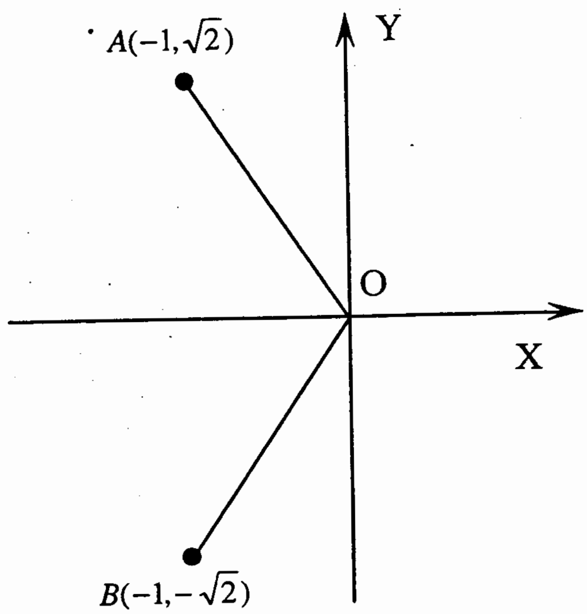
15 *Solution*

(a) Using the quadratic formula:

$x^2 + 2x + 3 = 0 \Rightarrow \Delta = -8 \Rightarrow x = \dfrac{-2 \pm i2\sqrt{2}}{2} = -1 \pm \sqrt{2}i$. Let $x_1 = -1 + \sqrt{2}i$ and

$x_2 = -1 - \sqrt{2}i$. Then $|x_1| = |x_2| = \sqrt{1+2} = \sqrt{3}$ and $\arg x_1 = \pi - \tan^{-1}\sqrt{2}$,

$\arg x_2 = -\left(\pi - \tan^{-1}\sqrt{2}\right)$.



(b) Using the quadratic formula:

$x^2 + 2px + q = 0 \Rightarrow \Delta = 4p^2 - 4q \Rightarrow x = \dfrac{-2p \pm i2\sqrt{q - p^2}}{2} = -p \pm \sqrt{q - p^2}\,i$, since

$p^2 < q$.

Let $x_3 = -p + i\sqrt{q - p^2}$ and $x_4 = -p - i\sqrt{q - p^2}$.

(i) Since $\angle HOK = 2\arg x_3$, if $p < 0$, or $\angle HOK = 2\pi - 2\arg x_3$, if $p > 0$,

$\angle HOK = \dfrac{\pi}{2} \Rightarrow \arg x_3 = \dfrac{\pi}{4}$ when $p < 0$ or $\arg x_3 = \dfrac{3\pi}{4}$ when $p > 0$. In each case

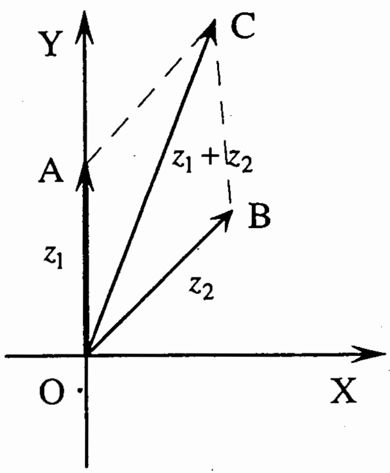
$\dfrac{\sqrt{q-p^2}}{|p|}$ must be equal to 1. Hence $\angle HOK$ is a right angle when $q - 2p^2 = 0$.

(ii) A, B, H and K will be equidistant from O, if $|x_1| = |x_2| = |x_3| = |x_4|$. But

$|x_1| = |x_2| = \sqrt{3}$ and $|x_3| = |x_4| = \sqrt{q}$. Hence $q = 3$.

Exercise 2.3

1 *Solution*



$\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. $OACB$ is a parallelogram and $\angle OC$ represents $z_1 + z_2$.

Since $|z_1| = 1$ and $|z_2| = 1$, $OA = OB$. Hence $OACB$ is a rhombus. Therefore $\angle COB = \frac{1}{2}\angle AOB$. But

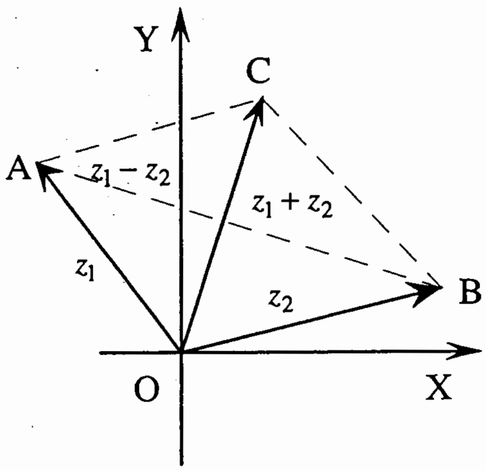
$\angle AOB = \frac{\pi}{2} - \arg z_2$ and $\angle COB = \arg(z_1 + z_2) - \arg z_2$.

Thus

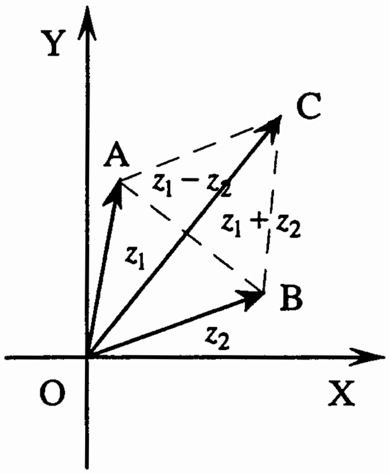
$\arg(z_1 + z_2) = \frac{1}{2}\left(\frac{\pi}{2} - \arg z_2\right) + \arg z_2 = \frac{\pi}{4} + \frac{1}{2}\arg z_2$.

Since $\arg z_2 = \frac{\pi}{4}$, $\arg(z_1 + z_2) = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$.

2 *Solution*

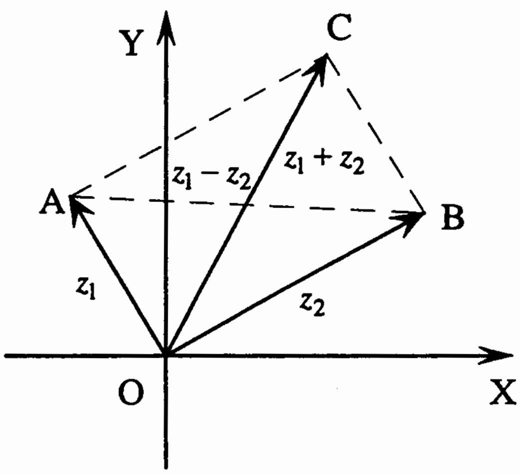


(a) Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. Construct the parallelogram $OACB$. Then $\overrightarrow{OC}$, $\overrightarrow{BA}$ represent $z_1 + z_2$, $z_1 - z_2$ respectively. Since $|z_1| = |z_2|$, $OA = OB$. Hence $OACB$ is a rhombus. Therefore diagonals OC and AB of $OACB$ meet at right angle. Thus $\overrightarrow{BA}$ is obtained from $\overrightarrow{OC}$ by a rotation anticlockwise (or clockwise) about O througl $\frac{\pi}{2}$, followed by an enlargement in O by some factor k, then by a translation to its position and a diagonal. Hence $z_1 - z_2 = ki(z_1 + z_2)$ (or $z_1 - z_2 = -ki(z_1 + z_2)$). In either case, the number $\dfrac{z_1 + z_2}{z_1 - z_2}$ is imaginary.



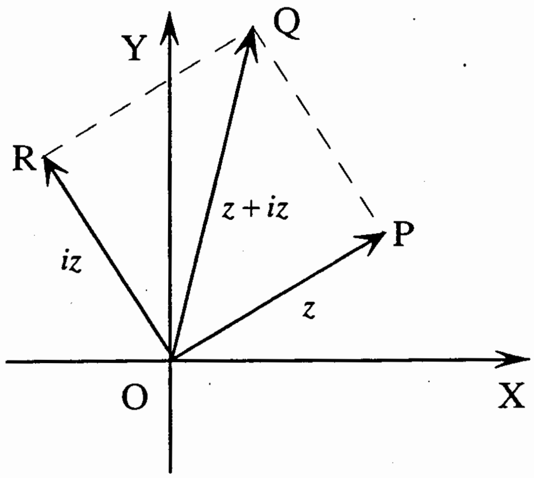
(b) Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. Construct the parallelogram $OACB$. Then $\overrightarrow{OC}$, $\overrightarrow{BA}$ represent z_1+z_2, z_1-z_2 respectively. Since $\arg(z_1-z_2)=\arg(z_1+z_2)+\frac{\pi}{2}$, $\overrightarrow{BA}$ is obtained from $\overrightarrow{OC}$ by a rotation anticlockwise about O through $\frac{\pi}{2}$, followed by an enlargement in O. Therefore diagonals OC and AB of the parallelogram $OACB$ meet at right angle and $OACB$ is a rhombus. Hence $OA=OB$ and $|z_1|=|z_2|$.

3 *Solution*



Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2. Construct the parallelogram $OACB$. Then $\overrightarrow{OC}$, $\overrightarrow{BA}$ represent z_1+z_2, z_1-z_2 respectively. Since $|z_1+z_2|=|z_1-z_2|$, $OC=AB$. Hence $OACB$ is a rectangular. Therefore $\angle AOB=\frac{\pi}{2}$. But $\angle AOB=\arg z_1-\arg z_2$ (or $\angle AOB=\arg z_2-\arg z_1$). Thus $\arg\left(\frac{z_1}{z_2}\right)=\pm\frac{\pi}{2}$.

4 Solution



Let R represent iz. We know that the transformation $z \to iz$ corresponds to a rotation anticlockwise about O through the angle $\frac{\pi}{2}$ in the Argand diagram. Therefore $OPQP$ is a square. Hence OPQ is a right-angled triangle.

5 Solution

$\overrightarrow{OP}$, $\overrightarrow{OQ}$ represent z_1, z_2. Since OPQ is an equilateral triangle, $OP = OQ$ and $\angle POQ = \frac{\pi}{3}$. Hence $\overrightarrow{OQ}$ is obtained from $\overrightarrow{OP}$ by a rotation anticlockwise (or clockwise) about O through $\frac{\pi}{3}$. Therefore $z_2 = \alpha z_1$ with $\alpha = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ (or $\alpha = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$).

$\therefore \quad z_1^2 + z_2^2 = z_1^2 \cdot \left(1+\alpha^2\right)$. But $1+\alpha^2 = \alpha$. Hence $z_1^2 + z_2^2 = \alpha z_1^2 = z_1 \cdot (\alpha z_1) = z_1 z_2$.

$\therefore \quad z_1^2 + z_2^2 = z_1 z_2$.

6 Solution

If $z_1 = 0$ or $z_2 = 0$, $\left||z_1| - |z_2|\right| = |z_1 + z_2|$. Let now $z_1 \neq 0$ and $z_2 \neq 0$. Then

$|z_1| - |z_2| = |z_1 + z_2 - z_2| - |z_2| \leq |z_1 + z_2| + |-z_2| - |z_2| = |z_1 + z_2|$ with equality if and only if $\quad z_1 + z_2 = k \cdot (-z_2)$, $k > 0$.

$\therefore \quad |z_1| - |z_2| \leq |z_1 + z_2|$ with equality if and only if $z_1 = -(1+k)z_2, k > 0$.

$|z_2| - |z_1| = |z_2 + z_1 - z_1| - |z_1| \leq |z_2 + z_1| + |-z_1| - |z_1| = |z_2 + z_1|$ with equality if and only if $\quad z_2 + z_1 = k \cdot (-z_1)$, $k > 0$.

$\therefore \quad |z_2| - |z_1| \leq |z_1 + z_2|$ with equality if and only if $z_1 = -\dfrac{1}{1+k} z_2, k > 0$.

Hence $\left||z_1| - |z_2|\right| \leq |z_1 + z_2|$ with equality if and only if $z_1 = -kz_2, k > 0$, or $z_1 = 0$, or $z_2 = 0$.

7 *Solution*

$|z_1+z_2| \le |z_1|+|z_2| = 25+6 = 31$ and this greatest value of 31 is attained when $z_2 = kz_1$ for some positive real k. But $|z_2| = 6$ and $z_2 = kz_1 \Rightarrow 6 = 25k$.

$\therefore |z_1+z_2|$ attained the greatest value of 31 when $z_2 = \frac{6}{25}(24+7i) = \frac{144}{25}+\frac{42}{25}i$.

$|z_1+z_2| \ge \big||z_1|+|z_2|\big| = 25-6 = 19$ and this least value of 19 is attained when $z_2 = -kz_1$ for some positive real k. But $|z_2| = 6$ and $z_2 = -kz_1 \Rightarrow 6 = 25k$.

$\therefore |z_1+z_2|$ attained the least value of 19 when $z_2 = -\frac{6}{25}(24+7i) = -\frac{144}{25}+\frac{42}{25}i$.

8 *Solution*

We shall use the method of mathematical induction to prove this inequality.

Define the statement $S(n)$: $|z_1+z_2+\cdots+z_n| \le |z_1|+|z_2|+\cdots+|z_n|$, $n = 2,3,\ldots$

Consider $S(2)$ $\quad |z_1+z_2| \le |z_1|+|z_2| \Rightarrow S(2)$ is true.

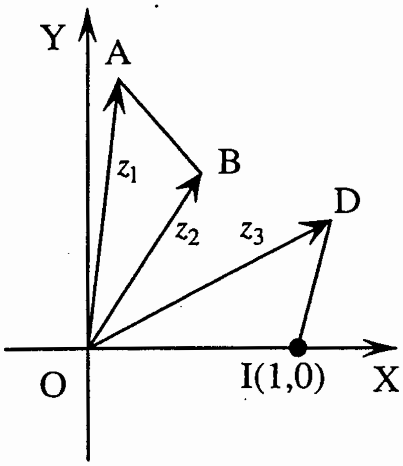
Let k be a positive integer, $k \ge 2$. If $S(k)$ is true, then

$|z_1+z_2+\cdots+z_k| \le |z_1|+|z_2|+\cdots+|z_k|$. Consider $S(k+1)$.

$|z_1+z_2+\cdots+z_k+z_{k+1}| \le |(z_1+z_2+\cdots+z_k)+z_{k+1}| \le |z_1+z_2+\cdots+z_k|+|z_{k+1}|$ (triangle inequality $S(2)$) $|z_1|+|z_2|+\cdots+|z_k|+|z_{k+1}|$, if $S(k)$ is true. Hence for all positive integers k $(k \ge 2)$, $S(k)$ true implies $S(k+1)$ true. But $S(2)$ is true, therefore by induction, $S(n)$ is true for all positive integers $n \ge 2$.

$\therefore$ $|z_1+z_2+\cdots+z_n| \le |z_1|+|z_2|+\cdots+|z_n|$, for all positive integers $n \ge 2$.

9 *Solution*



$\Delta OID \equiv \Delta OBA$,

$\therefore \dfrac{OD}{OA} = \dfrac{OI}{OB} \Rightarrow \dfrac{|z_3|}{|z_1|} = \dfrac{1}{|z_2|}$

$\angle DOI = \angle AOB \Rightarrow \arg z_3 = \arg z_1 - \arg z_2$. Hence

$|z_3| = \dfrac{|z_1|}{|z_2|}$ and $\arg z_3 = \arg z_1 - \arg z_2$

$\therefore z_3 = \dfrac{z_1}{z_2}$ and $\overrightarrow{OD}$ represents the quotient of z_1 and z_2.

Exercise 2.4

1 *Solution*

Let $z = 1 + i$. Then $z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ and

$1 - i = \bar{z} = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$.

$\therefore\ 1 \pm i = \sqrt{2}\,\mathrm{cis}\left(\pm\frac{\pi}{4}\right)$.

Using De Moivre's theorem $z^{20} = 2^{10}\,\mathrm{cis}(5\pi)$, $(\bar{z})^{20} = 2^{10}\,\mathrm{cis}(-5\pi)$. Now

$z^{20} + (\bar{z})^{20} = z^{20} + \overline{\left(z^{20}\right)} = 2\,\mathrm{Re}\left(z^{20}\right) = 2^{11}\cos(5\pi) = -2048$. Hence

$(1+i)^{20} + (1-i)^{20} = -2048$.

2 *Solution*

Let $z = -1 + \sqrt{3}i$. Then $z = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and

$-1 - \sqrt{3}i = \bar{z} = 2\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$. Using De Moivre's theorem

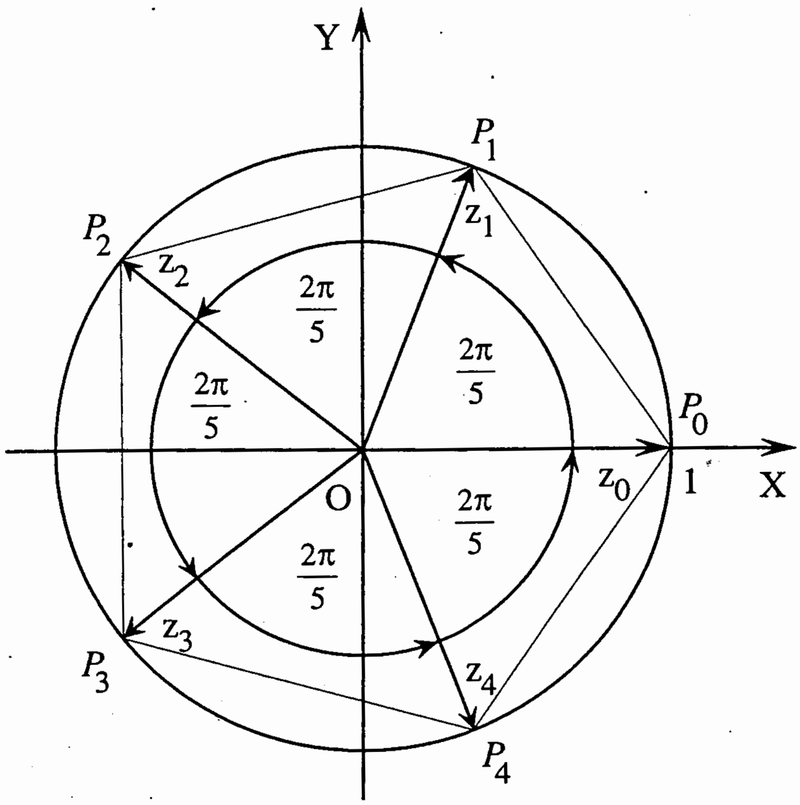
$z^n = 2^n\left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right)$. Now $z^n + (\bar{z})^n = z^n + \overline{\left(z^n\right)} = 2\,\mathrm{Re}\left(z^n\right) = 2^{n+1}\cos\left(\frac{2n\pi}{3}\right)$.

Thus $\left(-1+\sqrt{3}i\right)^n + \left(-1-\sqrt{3}i\right)^n = 2^{n+1}\cos\left(\frac{2n\pi}{3}\right)$.

If $n = 3m$, $\left(-1+\sqrt{3}i\right)^n + \left(-1-\sqrt{3}i\right)^n = 2^{n+1}\cos\left(\frac{6m\pi}{3}\right) = 2^{n+1}$.

If $n = 3m \pm 1$, $\left(-1+\sqrt{3}i\right)^n + \left(-1-\sqrt{3}i\right)^n = 2^{n+1}\cos\left(2m\pi \pm \frac{2\pi}{3}\right) = -2^n$.

3 *Solution*



$z^5 = 1 \Rightarrow |z| = 1$. Hence 5th roots of unity have modulus 1 and their representations P_k $(k = 0,1,2,3,4)$ lie on the unit circle with the centre in the origin. By De Moivre's theorem one root (z_0) has argument zero, the others being equally spaced around the unit circle in the Argand diagram by an angle $\frac{2\pi}{5}$. Hence the complex 5th roots of unity are 1, $\text{cis}\left(\pm\frac{2\pi}{5}\right)$, $\text{cis}\left(\pm\frac{4\pi}{5}\right)$.

Since $\angle P_k O P_{k+1} = \frac{2\pi}{5}$ and $OP_k = OP_{k+1} = 1$, $P_k P_{k+1} = 2\sin\frac{\pi}{5}$ for any $k = 0,1,2,3,4$ $(P_5 := P_0)$. Therefore the points P_k $(k = 0,1,2,3,4)$ form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}$ $\left(= 5\cdot(\text{area of } \Delta P_0 O P_1)\right)$ and perimeter $10\sin\frac{\pi}{5}$ $\left(= 5\cdot P_0 P_1\right)$.

4 *Solution*

$|-1| = 1$ and $\arg(-1) = \pi$. Hence the complex 5th roots of -1 all have modulus 1 and by De Moivre's theorem one complex 5th root of -1 has argument $\frac{\pi}{5}$, the others being equally spaced around the unit circle in the Argand diagram by an angle $\frac{2\pi}{5}$.

Therefore the complex 5th roots of -1 are $\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$, and -1.

Then $z^5 + 1 = (z+1)\left(z - \text{cis}\frac{\pi}{5}\right)\left(z - \text{cis}\left(-\frac{\pi}{5}\right)\right)\left(z - \text{cis}\frac{3\pi}{5}\right)\left(z - \text{cis}\left(-\frac{3\pi}{5}\right)\right)$. But

$\left(z - \text{cis}\frac{\pi}{5}\right)\left(z - \text{cis}\left(-\frac{\pi}{5}\right)\right) = \left(\left(z - \cos\frac{\pi}{5}\right) - i\sin\frac{\pi}{5}\right)\left(\left(z - \cos\frac{\pi}{5}\right) + i\sin\frac{\pi}{5}\right) = \left(z - \cos\frac{\pi}{5}\right)^2 + \left(\sin\frac{\pi}{5}\right.$

$= z^2 - 2z\cos\frac{\pi}{5} + 1$ and $\left(z - \text{cis}\frac{3\pi}{5}\right)\left(z - \text{cis}\left(-\frac{3\pi}{5}\right)\right) = z^2 - 2z\cos\frac{3\pi}{5} + 1$.

$\therefore\ z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$.

5 *Solution*

By De Moivre's theorem and $z^n = \cos n\theta + i\sin n\theta$ and $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta \; - i\sin n\theta$. Then $z^n + z^{-n} = 2\cos n\theta$ and $z^n - z^{-n} = 2i\sin n\theta$.

(a) $2\cos\theta = z + z^{-1}$. Then $16\cos^4\theta = \left(z + z^{-1}\right)^4$. But

$\left(z+z^{-1}\right)^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} \; = \left(z^4 + z^{-4}\right) + 4\left(z^2 + z^{-2}\right) + 6$. Hence

$16\cos^4\theta = 2\cos 4\theta + 4\cos 2\theta + 6$ and $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 2\cos 2\theta + 3)$.

(b) $2i\sin\theta = z - z^{-1}$. Then $32i^5\sin^5\theta = \left(z - z^{-1}\right)^5$. But

$\left(z - z^{-1}\right)^5 = z^5 - 5z^3 + 10z - 10z^{-1}$

$+5z^{-3} - z^{-5} = \left(z^5 - z^{-5}\right) - 5\left(z^3 - z^{-3}\right) + 10\left(z^1 - z^{-1}\right) = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$.

Hence $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$.

6 *Solution*

The cube roots of unity satisfy $z^3 - 1 = 0$. But $z^3 - 1 = (z-1)\left(z^2 + z + 1\right)$. Hence

(a) $z = 1 \Rightarrow z^2 + z + 1 = 3$

(b) $z \neq 1 \Rightarrow z^2 + z + 1 = 0$.

7 *Solution*

$\omega^3 = 1$. Since ω is a non-real root of unity, $\omega^2 + \omega + 1 = 0$ (it follows from the factorization $\omega^3 - 1 = (\omega - 1)\left(\omega^2 + \omega + 1\right)$).

Let $z_1 = \left(1 + 3\omega + \omega^2\right)^2$ and $z_2 = \left(1 + \omega + 3\omega^2\right)^2$.

$$\text{Then } z_1 = \left(1 + \omega + \omega^2 + 2\omega\right)^2 = (2\omega)^2 \quad \left(\text{since } 1 + \omega + \omega^2 = 0\right)$$

$$= 4\omega^2$$

$$\text{and } z_2 = \left(1 + \omega + \omega^2 + 2\omega^2\right)^2 = \left(2\omega^2\right)^2 \quad \left(\text{since } 1 + \omega + \omega^2 = 0\right)$$

$$= 4\omega^4 = 4\omega \qquad \left(\text{since } \omega^3 = 1\right)$$

Hence $z_1 + z_2 = 4\omega^2 + 4\omega = 4\left(\omega^2 + \omega + 1\right) - 4 = -4$ $\left(\text{since } \omega^2 + \omega + 1 = 0\right)$ and $z_1 \cdot z_2 = 4\omega^2 \cdot 4\omega = 16\omega^3 = 16$ $\left(\text{since } \omega^3 = 1\right)$.

8 *Solution*

(a) $z=\sqrt{3}+i=2\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)=2\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)$.

$\therefore |z|=2$ and $\arg z=\frac{\pi}{6}$.

By De Moivre's theorem one square root of z has modulus $\sqrt{2}$ and argument $\frac{\pi}{12}$.

Hence the two square roots of z are $\pm\sqrt{2}\operatorname{cis}\frac{\pi}{12}$.

(b) $z=-2-2i=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=8\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$.

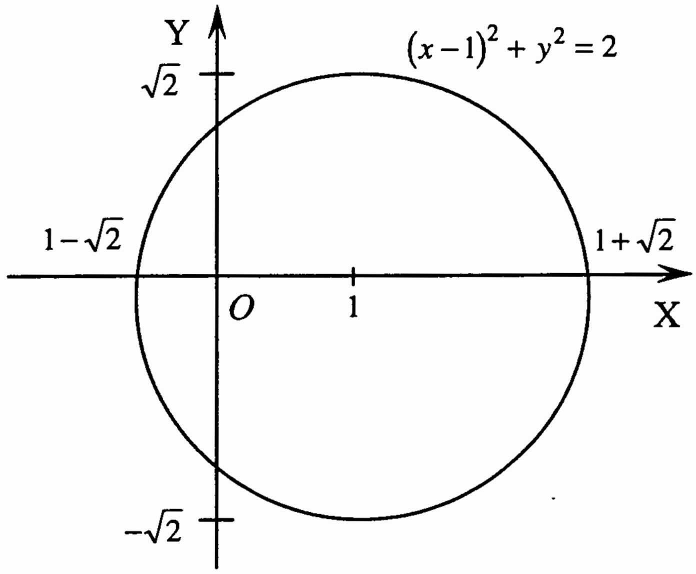
$\therefore |z|=\sqrt{8}$ and $\arg z=-\frac{3\pi}{4}$.

By De Moivre's theorem cube roots of z have modulus $\sqrt{2}$ and arguments $-\frac{\pi}{4}+\frac{2\pi k}{3}$, $k=-1,0,1$. Hence the three roots of z are

$\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$, $\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{12}\right)$, $\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{12}\right)$.

Exercise 2.5

1 *Solution*



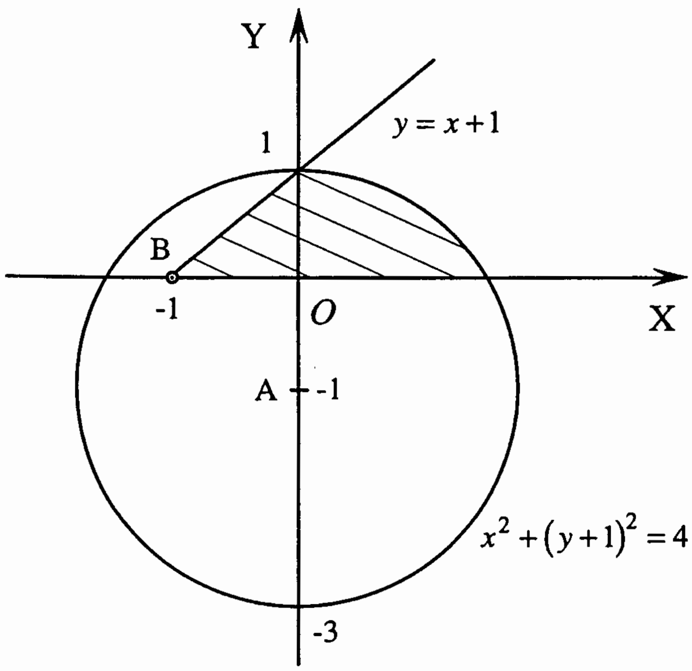
Let $z=x+iy$. Then $\bar{z}=x-iy$ and $|z|^2=x^2+y^2$,

$\therefore |z|^2=z+\bar{z}+1 \Leftrightarrow$

$x^2+y^2=2x+1 \Leftrightarrow$

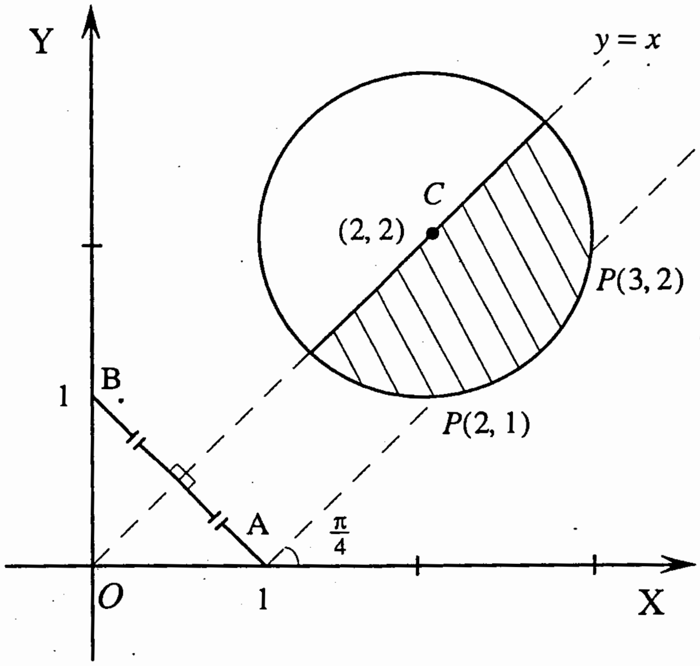
$(x-1)^2+y^2=2$. Hence P lies on the circle with centre $(1,0)$ and radius $\sqrt{2}$.

2 *Solution*



Let A represent $-i$ and B represent -1. Then, if P represents z, $\overrightarrow{AP}$ represents $z+i$ and $\overrightarrow{BP}$ represents $z+1$. Hence $AP \le 2$ and $\overrightarrow{BP}$ makes an angle between O and $\frac{\pi}{4}$ with the positive x-axis.

3 *Solution*

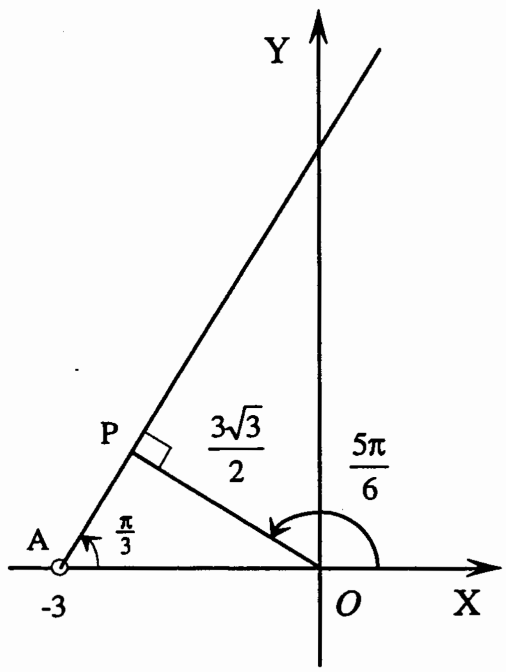


Let A,. B and Q represent 1, i, z respectively. If $|z-1|=|z-i|$, then $AQ=BQ$ and the locus of Q is the perpendicular bisector of AB. Since AB has midpoint $\left(\frac{1}{2},\frac{1}{2}\right)$ and gradient -1, the locus of Q passes through $\left(\frac{1}{2},\frac{1}{2}\right)$ with gradient 1 and has Cartesian equation $y=x$.

Let C represent $2+2i$. If $|z-2-2i|\le 1$, then $CQ=1$ and Q lies on or inside the circle with centre $(2,2)$ and radius 1.

Let now $|z-1|\le|z-i|$ and $|z-2-2i|\le 1$. Then $AQ\le BQ$ and $CQ\le 1$. Hence Q lies on the right-hand side of the perpendicular bisector of AB inside the circle centre C and radius 1, or Q lies on the boundary of this region. If P describes the boundary of this region and $\arg(z-1)=\frac{\pi}{4}$, then $CP=1$ and $\overrightarrow{AP}$ makes the angle $\frac{\pi}{4}$ with the positive x-axis. Thus we must solve simultaneously two Cartesian equations $(x-2)^2+(y-2)^2=1$ and $y=x-1$. Substituting the second equation into the first gives $(x-2)^2+(x-3)^2=1 \Rightarrow 2x^2-10x+12=0 \Rightarrow x=2,3 \Rightarrow y=1$ (when $x=2$), $y=2$ (when $x=3$). Therefore such P represents $z=2+i$ and $z=3+2i$.

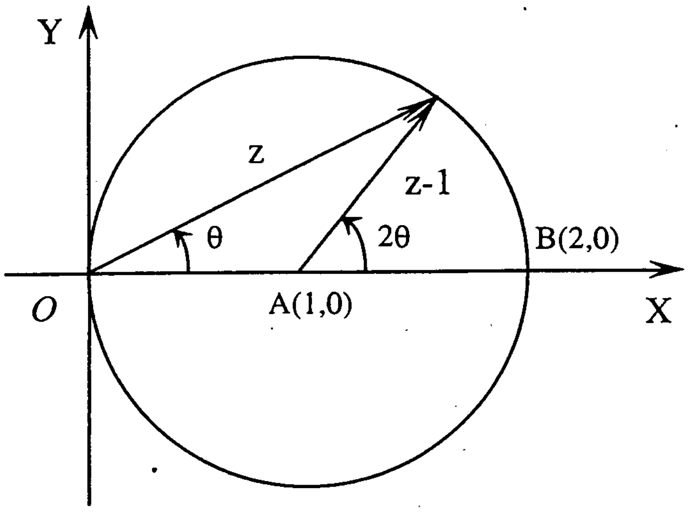
4 *Solution*



Let A represent -3. Then $\overrightarrow{AP}$ represents $z+3$. AP has gradient $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$. Hence the locus of P has Cartesian equation $y=\sqrt{3}x+3\sqrt{3}, x>-3$. Now $OP=|z|$. Hence the minimum value of $|z|$ is the perpendicular distance from $(0,0)$ to the locus of P. Therefore the minimum value of $|z|$ is $AO\cdot\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}$. Since AP has gradient $\tan\frac{\pi}{3}=\sqrt{3}$, OP has gradient $-\frac{1}{\sqrt{3}}=\tan\left(\frac{5\pi}{6}\right)$ when $|z|$ takes its least value. Hence modulus of z is $\frac{3\sqrt{3}}{2}$ and the argument of z is $\frac{5\pi}{6}$ when $|z|$ is a minimum. Therefore

$$z=\frac{3\sqrt{3}}{2}\operatorname{cis}\left(\frac{5\pi}{6}\right)=\frac{3\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=\frac{3}{4}\left(-3+i\sqrt{3}\right).$$

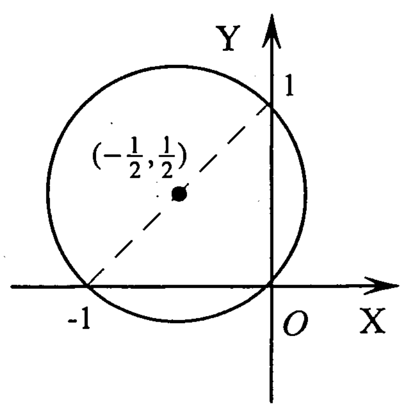
5 *Solution*



Let A represent 1. Then $\overrightarrow{AP}$ represents $z-1$ and $AP=1$. Hence P lies on the circle centre $A(1,0)$ and radius 1. Let $\theta=\arg z$ and B represent 2. Then $\angle POB=\theta$ and $\angle PAB=\arg(z-1)$. But $\angle PAB=2\angle POB$ and $\arg\left(z^2\right)=2\arg z$. Therefore

$\arg(z-1)=2\theta=2\arg z=\arg\left(z^2\right)$.

6 *Solution*



Let $P(x,y)$ represent $z=x+iy$. Then

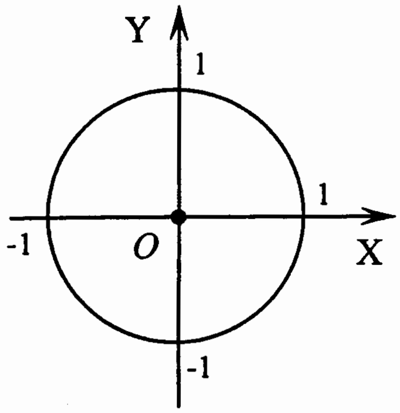
$$\frac{z-i}{z+1}=\frac{x+i(y-1)}{(x+1)+iy}=\frac{(x+i(y-1))((x+1)-iy)}{(x+1)^2+y^2}=$$

$$\frac{x(x+1)+y(y-1)+i((y-1)(x+1)-xy)}{(x+1)^2+y^2},$$

$\therefore$ if $\dfrac{z-i}{z+1}$ is purely imaginary, then

$x(x+1)+y(y-1)=0$. This is the equation of the circle with centre $\left(-\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$.

7 *Solution*



Let $P(x, y)$ represent $z = x + iy$. Then

$$z - \frac{1}{z} = x + iy - \frac{1}{x+iy} = x + iy - \frac{x-iy}{x^2+y^2} =$$

$$\left(x - \frac{x}{x^2+y^2}\right) + i\left(y + \frac{y}{x^2+y^2}\right)$$. Hence, if

$\operatorname{Re}\left(z - \frac{1}{z}\right) = 0$, then $x - \frac{x}{x^2+y^2} = 0$.

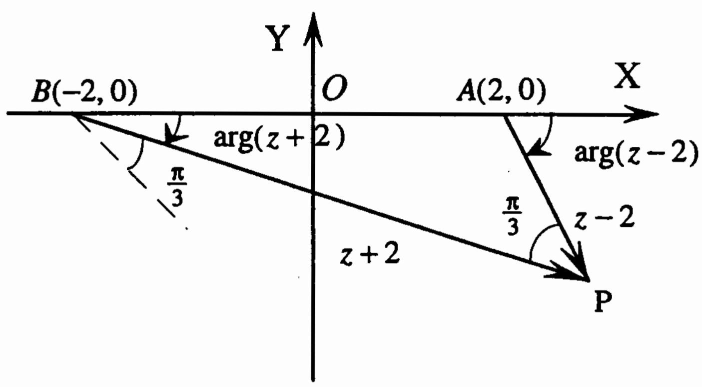
$$\therefore\ x = 0,\ \ 1 - \frac{1}{x^2+y^2} = 0.$$

Therefore the locus of the point P has Cartesian equation $x = 0\,(y \neq 0)$ or

$x^2 + y^2 = 1$.

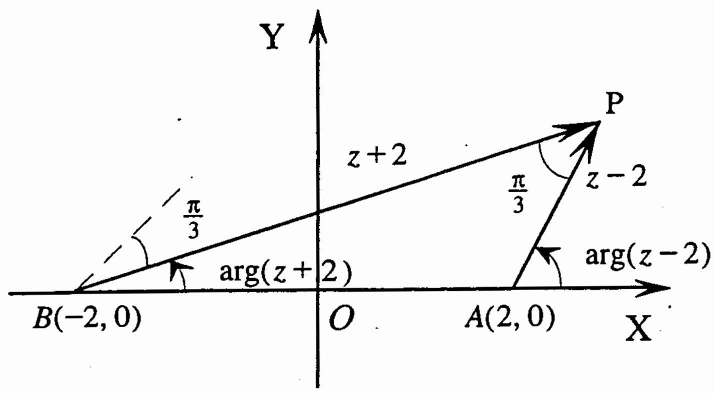
8 *Solution*

Let $A(2,0)$, $B(-2,0)$ and P represent 2, -2, and z respectively. Then $\overrightarrow{AP}$ and $\overrightarrow{BP}$ represent $z-2$ and $z+2$ respectively, and $\arg(z-2) = \arg(z+2) + \frac{\pi}{3}$ requires $\overrightarrow{AP}$ to be parallel to the vector obtained by rotation of $\overrightarrow{BP}$ anticlockwise through the angle of $\frac{\pi}{3}$.

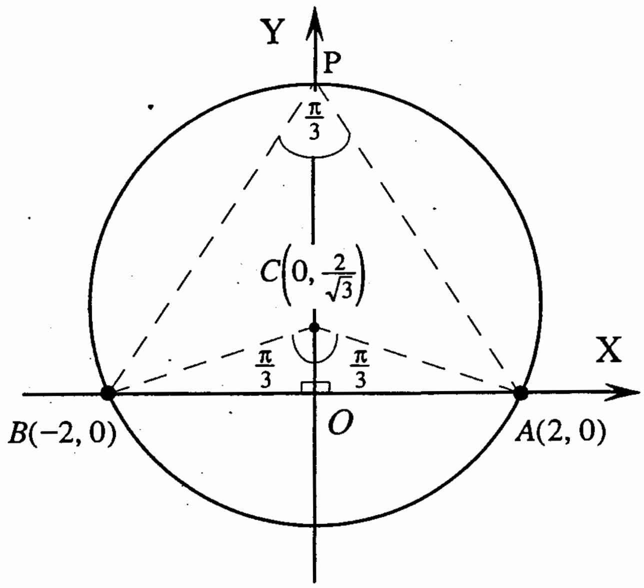


If P lies below the x-axis, AP must be parallel to a clockwise rotation of BP. This diagram shows $\arg(z-2) = \arg(z+2) - \frac{\pi}{3}$.

Hence P must lie above the x-axis.



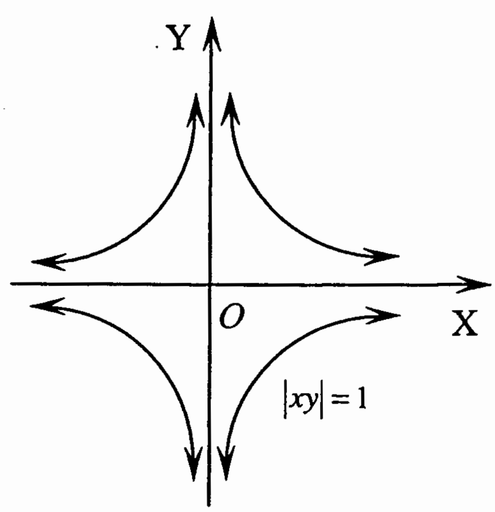
Since alternate angles between parallel lines are equal, $\angle BPA = \frac{\pi}{3}$ as P traces its locus. Hence P lies on the major arc AB of a circle through A and B.



The centre C of this circle lies on the perpendicular bisector of AB, and the chord AB subtends an angle $2 \cdot \frac{\pi}{3} = \frac{2\pi}{3}$ at C.

Therefore $OC = \frac{2}{\sqrt{3}}$ and $AC = \frac{4}{\sqrt{3}}$. Thus the centre of this circle is $C\left(0, \frac{2}{\sqrt{3}}\right)$ and the is radius $\frac{4}{\sqrt{3}}$.

9 *Solution*



Let $P(x, y)$ represent $z = x + iy$. Then

$z^2 - \bar{z}^2 = (z - \bar{z})(z + \bar{z}) = (2iy) \cdot (2x)$,

$\therefore \left|z^2 - \bar{z}^2\right| = 4|xy|$,

$\therefore |xy| = 1$.

10 *Solution*

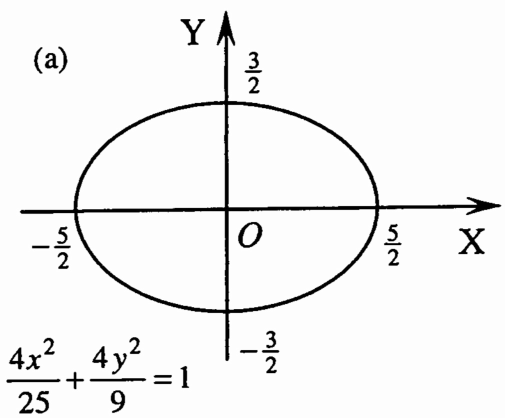
Let $P(x, y)$ represent $z = x + iy$. Then $x + iy = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) =$

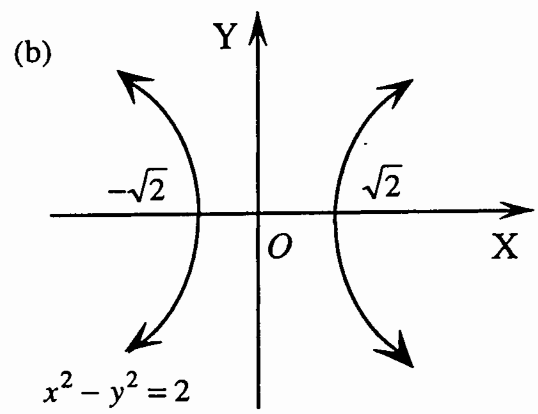
$\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$,

$\therefore x = \left(r + \frac{1}{r}\right)\cos\theta$ and $y = \left(r - \frac{1}{r}\right)\sin\theta$.

(a) $x = \frac{5}{2}\cos\theta$ and $y = \frac{3}{2}\sin\theta$. Hence $\dfrac{4x^2}{25} + \dfrac{4y^2}{9} = 1$.

(b) $x = \left(r + \frac{1}{r}\right)\frac{1}{\sqrt{2}}$ and $y = \left(r - \frac{1}{r}\right)\frac{1}{\sqrt{2}}$. Hence $x^2 - y^2 = 2$.





Diagnostic Test 2

1 *Solution*

(a) (i) $z_1 + z_2 = (2+i) + i = 2 + 2i$

(ii) $z_1 + z_2 = (4+i) + (2+3i) = 6 + 4i$

(b) (i) $z_1 - z_2 = (2+i) - i = 2$

(ii) $z_1 - z_2 = (4+i) - (2+3i) = 2 - 2i$

(c) (i) $z_1 z_2 = (2+i)i = 2i + i^2 = -1 + 2i$

(ii) $z_1 z_2 = (4+i)\cdot(2+3i) = 8 + 3i^2 + 12i + 2i = 5 + 14i$

(d) (i) $\dfrac{z_1}{z_2} = \dfrac{2+i}{i} = \dfrac{(2+i)(-i)}{i\cdot(-i)} = \dfrac{1-2i}{1} = 1 - 2i$

(ii) $\dfrac{z_1}{z_2} = \dfrac{4+i}{2+3i} = \dfrac{(4+i)(2-3i)}{(2+3i)(2-3i)} = \dfrac{(8+3)+(2-12)i}{4+9} = \dfrac{11}{13} - \dfrac{10}{13}i$

2 *Solution*

(a) (i) $\mathrm{Re}(3) = 3$ (ii) $\mathrm{Re}(4i) = 0$ (iii) $\mathrm{Re}(3+4i) = 3$

(b) (i) $\mathrm{Im}(3) = 0$ (ii) $\mathrm{Im}(4i) = 4$ (iii) $\mathrm{Im}(3+4i) = 4$

(c) (i) $\overline{(3)} = 3$ (ii) $\overline{(4i)} = -4i$ (iii) $\overline{(3+4i)} = 3 - 4i$

3 *Solution*

$(x+iy)^2 = 3 + 4i \Rightarrow (x^2 - y^2) + (2xy)i = 3 + 4i$

Equating real and imaginary parts: $x^2 - y^2 = 3$ and $2xy = 4$

$\therefore\ x^4 - x^2y^2 = 3x^2$ and $x^2y^2 = 4$

Then $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2-4)(x^2+1) = 0$, x real,

$\therefore\ x = 2,\ y = 1$ or $x = -2,\ y = -1$.

4 *Solution*

(a) $\Delta = -4 = 4i^2 \Rightarrow x = \dfrac{-2 \pm 2i}{2} = -1 \pm i$

(b) Find Δ: $\Delta = (2-i)^2 + 8i = 3 + 4i$.

Find square roots of Δ: Let $(a+ib)^2 = 3+4i, \quad a,b \in \mathbf{R}$.

Then $a^2 - b^2 = 3$ and $2ab = 4$.

$\therefore \quad a^4 - a^2b^2 = 3a^2$ and $a^2b^2 = 4$.

Thus $a^4 - 3a^2 - 4 = 0 \Rightarrow (a^2-4)(a^2+1) = 0$, a real,

$\therefore \quad a = 2, b = 1$ or $a = -2, b = -1$.

Hence Δ has the square roots $2+i, \quad -2-i$.

Use the quadratic formula: $x^2 + (2-i)x - 2i = 0$

has solutions $x = \dfrac{-(2-i) \pm (2+i)}{2}$

$\therefore \quad x = i$ or $x = -2$.

5 *Solution*

(a) $z = 2 \cdot (\cos 0 + i \sin 0) \Rightarrow |z| = 2, \ \arg z = 0$

(b) $z = 2i = 2\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) \Rightarrow |z| = 2, \quad \arg z = \frac{\pi}{2}$

(c) $z = 1 + \sqrt{3}i = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \Rightarrow |z| = 2, \quad \arg z = \frac{\pi}{3}$

(d) $z = -\sqrt{3} - i = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) \Rightarrow |z| = 2, \quad \arg z = -\frac{5\pi}{6}$.

6 *Solution*

(a) $z = -1 + i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

(b) $z = 1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$.

7 Solution

(a) $z = 4\text{cis}\left(\frac{2\pi}{3}\right) = 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2 + i2\sqrt{3}$

(b) $z = 2\text{cis}\left(-\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i$

8 Solution

$|z_1| = 2$ and $\arg z_1 = \frac{\pi}{3}$, $|z_2| = \sqrt{2}$ and $\arg z_2 = -\frac{\pi}{4}$.

(a) $|z_1 z_2| = |z_1| \cdot |z_2| = 2\sqrt{2}$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$.

(b) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{7\pi}{12}.$$

9 Solution

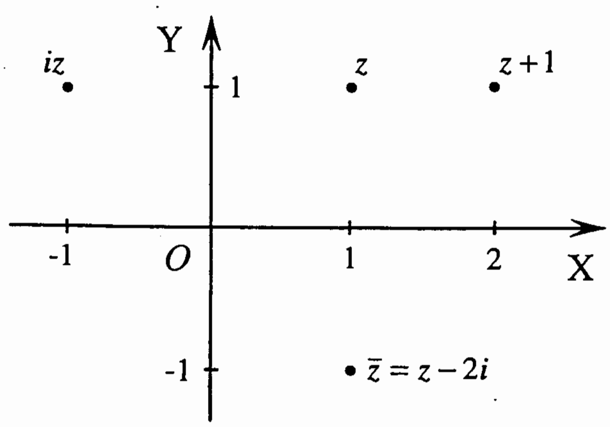
$z = 1 + i = \sqrt{2} \cdot \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\text{cis}\frac{\pi}{4}$,

$\therefore$ $|z| = \sqrt{2}$ and $\arg z = \frac{\pi}{4}$. Then $|z^{10}| = |z|^{10} = \left(\sqrt{2}\right)^{10} = 32$,

$\arg\left(z^{10}\right) = 10\arg z = 10 \cdot \frac{\pi}{4} = \frac{5\pi}{2}$. But $\frac{5\pi}{2} > \pi$. The principal argument of z^{10} is

$\frac{5\pi}{2} - 2\pi = \frac{\pi}{2}$.

10 Solution



(a) $z = 1 + i$

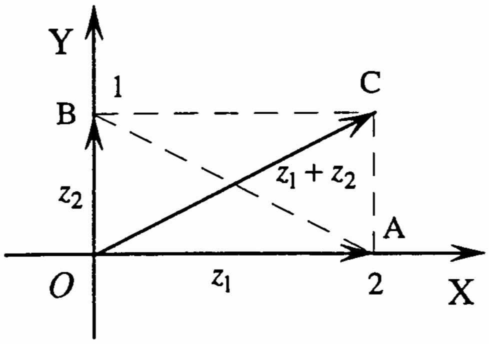
(b) $\bar{z} = 1 - i$

(c) $iz = i + i^2 = -1 + i$

(d) $z + 1 = 2 + i$

(e) $z - 2i = 1 - i$

11 *Solution*



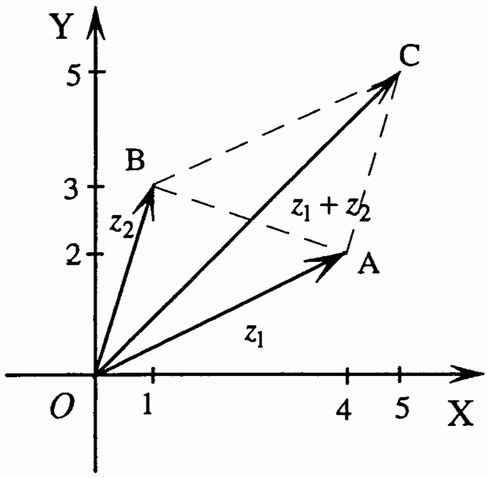
(i) Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent z_1, z_2.

Then (a) $\overrightarrow{OC}$ represents $z_1 + z_2$

(b) $\overrightarrow{BA}$ represents $z_1 - z_2$

(c) $\overrightarrow{AB}$ represents $z_2 - z_1$.

(ii)



12 *Solution*

By De Moivre's theorem: $(\cos\theta + i\sin\theta)^4 = \cos(4\theta) + i\sin(4\theta) = \mathrm{cis}(4\theta)$.

13 *Solution*

By De Moivre's theorem: $\cos(2\theta) - i\sin(2\theta) = (\cos\theta + i\sin\theta)^{-2} = (\mathrm{cis}\theta)^{-2}$.

14 *Solution*

By De Moivre's theorem: $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$. But

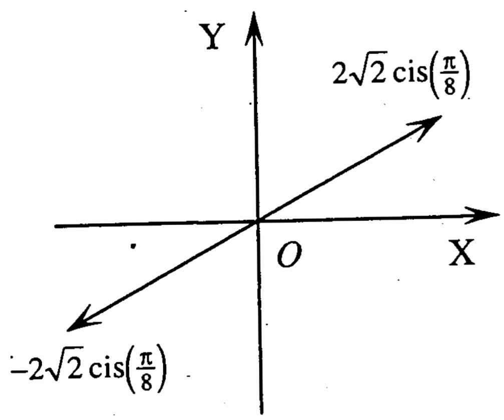
$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta = \left(\cos^2\theta - \sin^2\theta\right) + i2\sin\theta\cos\theta$.

Equating real and imaginary parts we obtain $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and

$\sin(2\theta) = 2\sin\theta\cos\theta$. Hence

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{\cos^2\theta \cdot 2\dfrac{\sin\theta}{\cos\theta}}{\cos^2\theta \cdot \left(1 - \dfrac{\sin^2\theta}{\cos^2\theta}\right)} = \frac{2\tan\theta}{1-\tan^2\theta}.$$

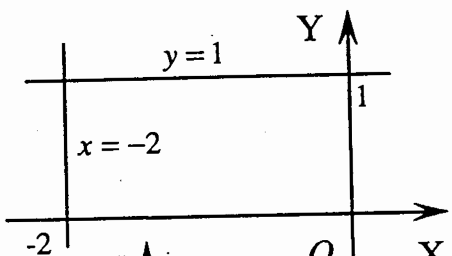
15 *Solution*



$z = 8\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 8\mathrm{cis}\frac{\pi}{4}$,

$\therefore$ $|z| = 8$ and $\arg z = \frac{\pi}{4}$. By De Moivre's theorem, one square root of z has modulus $2\sqrt{2}$ and argument $\frac{\pi}{8}$. Hence the two square roots of z are $\pm 2\sqrt{2}\,\mathrm{cis}\left(\frac{\pi}{8}\right)$.

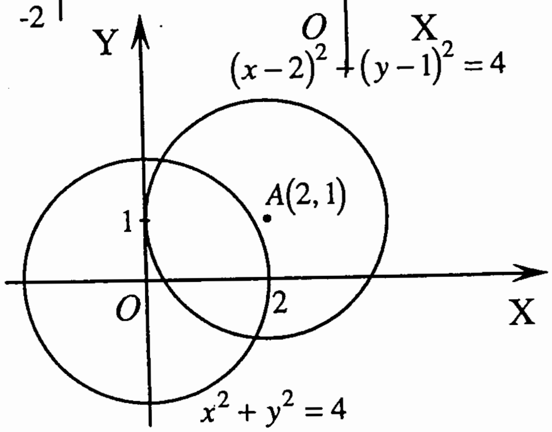
16 *Solution*



Let $z = x + iy$. Then

(a) $\mathrm{Re}\, z = -2 \Rightarrow x = -2$,

(b) $\mathrm{Im}\, z = 1 \Rightarrow y = 1$.

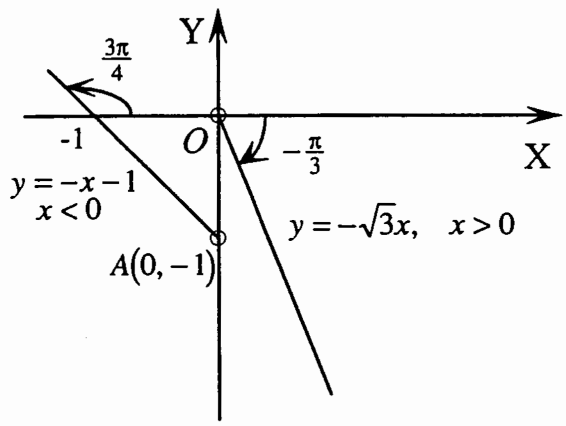


Let P represent z. Then

(c) $OP = |z|$. $|z| = 2 \Rightarrow P$ lies on the circle, center $(0,0)$ and radius 2.

(d) Let A represent $2+i$. Then $\overrightarrow{AP}$ represents $z-(2+i)$ and $|z-2-i| = 2 \Rightarrow AP = 2$,

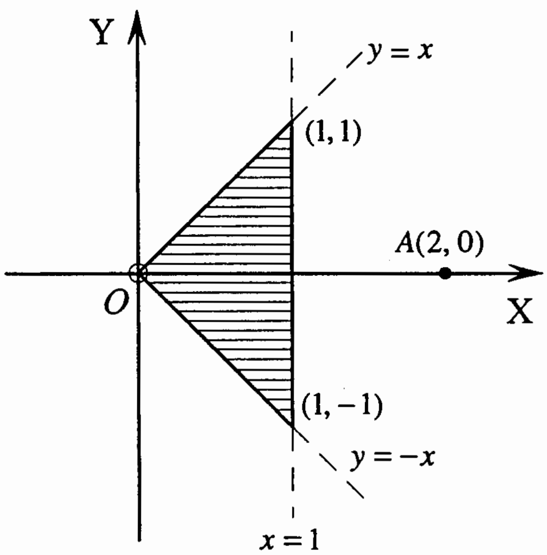
$\therefore$ P lies on the circle with the centre $A(2,1)$ and radius 2.



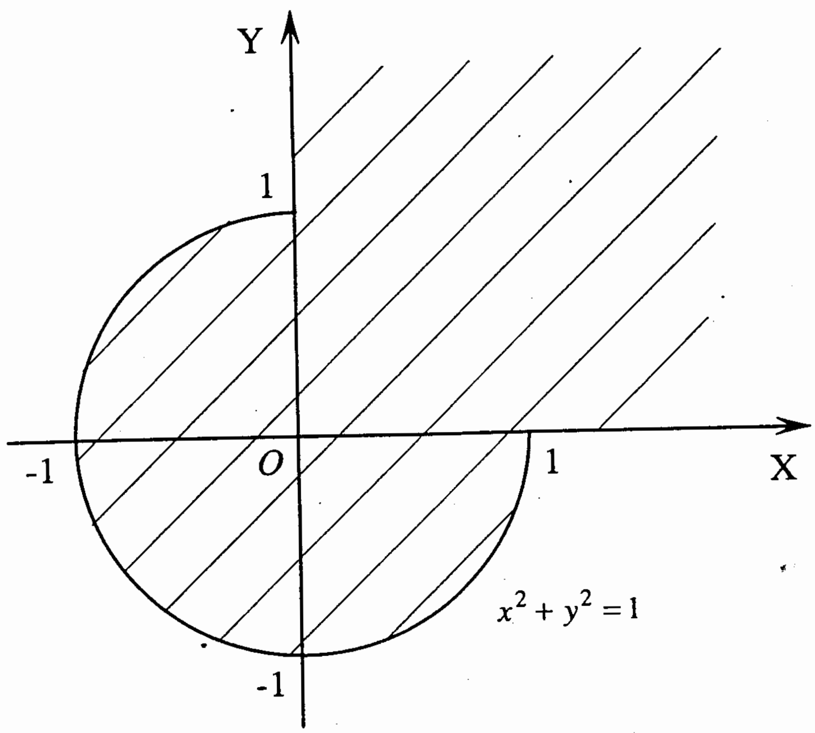
(e) The gradient of $\overrightarrow{OP}$ is $\tan\left(-\frac{\pi}{3}\right)=-\sqrt{3}$. The locus P is the ray $y=-\sqrt{3}x, \quad x>0$.

(f) Let A represent $-i$. Then $\overrightarrow{AP}$ represents $z+i$. AP has gradient $\tan\left(\frac{3\pi}{4}\right)=-1$. Hence the locus of P is the ray $y=-x-1$, $x<0$.

17 *Solution*



(a) $|z|=|z-2|$ is the perpendicular bisector of OA. $\arg z=\frac{\pi}{4}$ is the ray $y=x$, $x>0$. $\arg z=-\frac{\pi}{4}$ is the ray $y=-x$, $x>0$.



(b) $|z| = 1$ is the circle, centre $(0,0)$ and radius 1. $\arg z = 0$ is the positive x-axis. $\arg z = \frac{\pi}{2}$ is the positive y-axis.

Further Questions 2

1 *Solution*

Let $z_1 = 3+2i$ and $z_2 = 5+4i$. Then

$z_1z_2 = (3+2i)(5+4i) = (15-8)+i(12+10) = 7+22i$,

$\bar{z}_1\bar{z}_2 = (3-2i)(5-4i) = (15-8)-i(12+10) = 7-22i$.

Hence $|z_1z_2|^2 = 7^2+22^2$. But $|z_1z_2|^2 = |z_1|^2\cdot|z_2|^2 = \left(3^2+2^2\right)\left(5^2+4^2\right)$. Therefore

$7^2+22^2 = \left(3^2+2^2\right)\left(5^2+4^2\right)$.

2 *Solution*

$$z_1 = \frac{a}{1+i} = \frac{a(1-i)}{(1+i)(1-i)} = \frac{a-ia}{1+1} = \frac{a}{2} - \frac{a}{2}i,$$

$$z_2 = \frac{b}{1+2i} = \frac{b(1-2i)}{(1+2i)(1-2i)} = \frac{b-2ib}{1+4} = \frac{b}{5} - \frac{2b}{5}i.$$

Hence $z_1+z_2 = \left(\frac{a}{2}+\frac{b}{5}\right) - i\left(\frac{a}{2}+\frac{2b}{5}\right)$. But $z_1+z_2 = 1$ and a, b are real. Equating real and imaginary parts:

$\frac{a}{2}+\frac{b}{5} = 1$ and $\frac{a}{2}+\frac{2b}{5} = 0$. Therefore $a = 4$, $b = -5$

3 *Solution*

Substituting $x = 1+i$, $(1+i)^2 + (a+2i)(1+i) + (5+ib) = 0$,

$\therefore\ \ (1-1)+2i+(a-2)+i(a+2)+5+ib = 0$,

$\therefore\ \ (a+3)+i(a+b+4) = 0,\ \ a, b \in \mathbf{R}$.

Equating real and imaginary parts: $a+3=0$ and $a+b+4=0$.
Therefore $a=-3$, $b=-1$.

4 *Solution*

Let z be the other root of the equation $x^2+(1+i)x+k=0$. Then $z+(1-2i)=-(1+i)$ and $z\cdot(1-2i)=k$. Therefore $z=-(1+i)-(1-2i)=-2+i$ and $k=(-2+i)(1-2i)=(-2+2)+i(4+1)=5i$. Hence $k=5i$ and equation $x^2+(1+i)x+k=0$ has roots $x=1-2i$ and $x=-2+i$.

5 *Solution*

Let z_1, z_2 are the roots of the equation $x^2+(a+ib)x+3i=0$. Then $z_1^2+(a+ib)z_1+3i=0$ and $z_2^2+(a+ib)z_2+3i=0$. But $z_1^2+z_2^2=8$. Hence $8+(a+ib)(z_1+z_2)+6i=0$. But $z_1+z_2=-(a+ib)$. Therefore $8-(a+ib)^2+6i=0$,

$\therefore\quad (a+ib)^2=8+6i,\quad a,b\in\mathbf{R}$.

Thus $(a^2-b^2)+(2ab)i=8+6i$. Equating real and imaginary parts, $a^2-b^2=8$ and $2ab=6$. $a^2-\dfrac{9}{a^2}=8\Rightarrow a^4-8a^2-9=0$. $(a^2-9)(a^2+1)=0$, a real.

$\therefore\quad a=3,\quad b=1$ or $a=-3,\quad b=-1$.

6 *Solution*

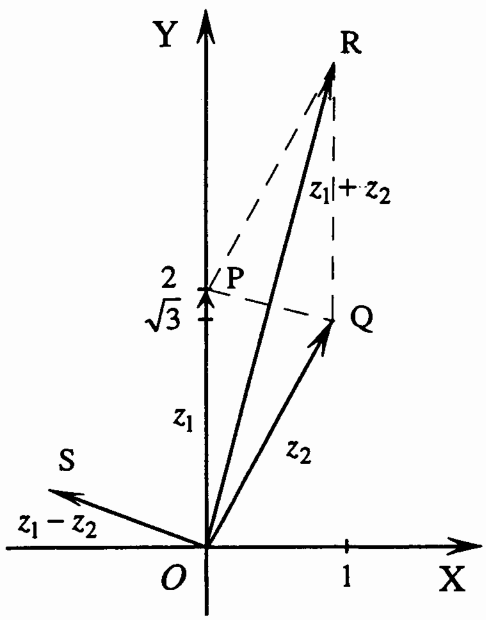
Find Δ: $\Delta=4^2-4(1-4i)=12+16i$.

Find square roots of Δ: Let $(a+ib)^2=12+16i,\quad a,b\in\mathbf{R}$. Then $(a^2-b^2)+(2ab)i=12+16i$. Equating real and imaginary parts, $a^2-b^2=12$ and $2ab=16$. $a^2-\dfrac{64}{a^2}=12\Rightarrow a^4-12a^2-64=0$, $(a^2-16)(a^2+4)=0$, a real.

$\therefore\quad a=4,\quad b=2$ or $a=-4,\quad b=-2$. Hence Δ has square roots $4+2i$, $-4-2i$. Use the quadratic formula: $x^2-4x+(1-4i)=0$ has the solutions $x=\dfrac{4\pm(4+2i)}{2}$,

$\therefore\quad x=-i$ or $x=4+i$.

7 *Solution*



$z_1 = 2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$,

$\therefore \quad |z_1| = 2$ and $\arg z_1 = \frac{\pi}{2}$.

$z_2 = 1 + \sqrt{3}i = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$,

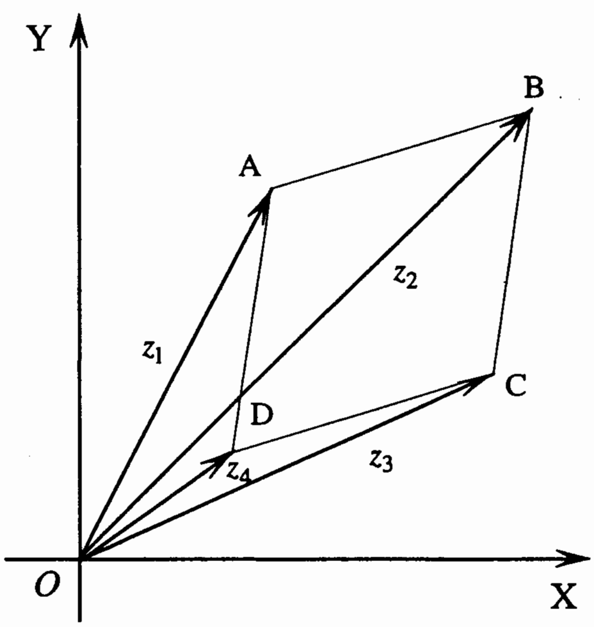
$\therefore \quad |z_2| = 2$ and $\arg z_2 = \frac{\pi}{3}$.

$OP = |z_1|$, $OQ = |z_2|$. But $|z_1| = |z_2|$. Hence $OP = OQ$ and $OPRQ$ is a rhombus. Therefore $\angle POR = \angle QOR$. Thus $\arg(z_1 + z_2) = \frac{1}{2}(\arg z_1 + \arg z_2) = \frac{5\pi}{12}$. Since diagonals OR and QP of the rhombus $OPRQ$ meet at right angle,

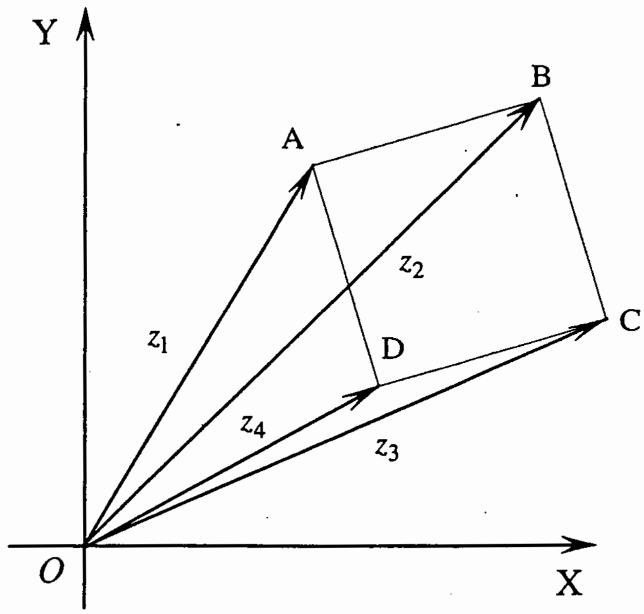
$\arg(z_1 - z_2) = \arg(z_1 + z_2) + \frac{\pi}{2} = \frac{11\pi}{12}$.

$\therefore \quad \arg(z_1 + z_2) = \frac{5\pi}{12}$, $\arg(z_1 - z_2) = \frac{11\pi}{12}$.

8 *Solution*



If $z_1 - z_2 + z_3 - z_4 = 0$, then $z_1 - z_2 = z_4 - z_3$. But $\overrightarrow{BA}$ represents $z_1 - z_2$, $\overrightarrow{CD}$ represents $z_4 - z_3$. Therefore $\overrightarrow{BA}$ and $\overrightarrow{CD}$ are parallel. On the other hand, $z_1 - z_4 = z_2 - z_3$. But $\overrightarrow{DA}$ represents $z_1 - z_4$, $\overrightarrow{CB}$ represents $z_2 - z_3$. Hence $\overrightarrow{DA}$ and $\overrightarrow{CB}$ are parallel. So we proved that $ABCD$ is a parallelogram.



If $z_1 + iz_2 - z_3 - iz_4 = 0$, then $z_1 - z_3 = i(z_4 - z_2)$. Hence the diagonals CA and BD of the parallelogram $ABCD$ meet at right angle and $CA = BD$. Therefore $ABCD$ is a square.

9 *Solution*

Noting $r^2 = z\bar{z}$, $\dfrac{z}{z^2 + r^2} = \dfrac{z}{z^2 + z\bar{z}} = \dfrac{1}{z + \bar{z}} = \dfrac{1}{2\operatorname{Re} z}$. Hence $\dfrac{z}{z^2 + r^2}$ is real. Since $\operatorname{Re} z = r\cos\theta$, $\dfrac{z}{z^2 + r^2} = \dfrac{1}{2r\cos\theta}$.

10 *Solution*

The cube roots of unity satisfy $x^3 - 1 = 0$. But $x^3 - 1 = (x-1)(x^2 + x + 1)$. Hence $\omega \neq 1 \Rightarrow \omega^2 + \omega + 1 = 0$. Clearly, $\omega^3 = 1$. Therefore $(1 + \omega^2)^{12} = (-\omega)^{12} = (\omega^3)^4 = 1$.

Then

$$\begin{aligned}
\omega^4 &= \omega^3 \cdot \omega = \omega, \\
\omega^5 &= \omega^3 \cdot \omega^2 = \omega^2, \\
\omega^7 &= \omega^6 \cdot \omega = \omega, \\
\omega^8 &= \omega^6 \cdot \omega^2 = \omega^2.
\end{aligned}$$

Hence

$$(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)(1-\omega^7)(1-\omega^8) =$$

$$\left((1-\omega)(1-\omega^2)\right)^3 = \left(1 - \omega - \omega^2 + \omega^3\right)^3 = \left(2 - \omega - \omega^2\right)^3 =$$

$$\left(3 - (1 + \omega + \omega^2)\right)^3 = 3^3 = 27.$$

11 *Solution*

The cube roots of unity satisfy $z^3 - 1 = 0$. Therefore, if z is a common root of the equations $z^3 - 1 = 0$ and $pz^5 + qz + r = 0$, then z is one of the cube roots. Thus
if $z = 1$, then $p + q + z = 0$;
if $z = \omega$, then $p\omega^5 + q\omega + r = 0$;
if $z = \omega^2$, then $p\omega^{10} + q\omega^2 + r = 0$.
Hence $(p + q + r)(p\omega^5 + q\omega + r)(p\omega^{10} + q\omega^2 + r) = 0$.

12 *Solution*

$z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$. Therefore, if $z^6 + z^3 + 1 = 0$, then $z^9 - 1 = 0$. Hence the roots of $z^6 + z^3 + 1 = 0$ are among the roots of $z^9 - 1 = 0$. Let $z = \cos\theta + i\sin\theta$ satisfy $z^9 = 1$. Using De Moivre's theorem, $\cos(9\theta) + i\sin(9\theta) = 1 + 0i$

$\therefore\ \cos(9\theta) = 1$ and $\sin(9\theta) = 0$

$\therefore\ 9\theta = 2\pi k$, k integral.

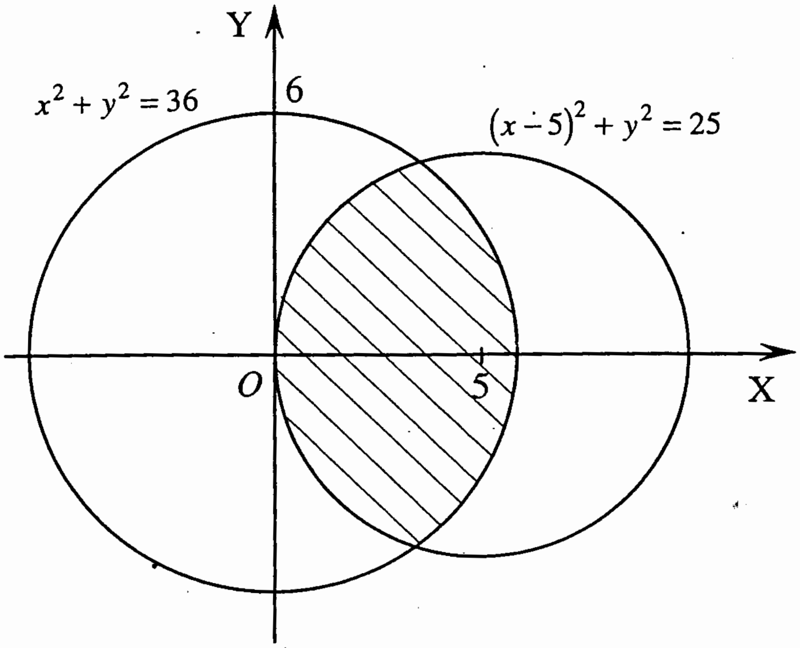
$\therefore\ \theta = \frac{2\pi}{9}k$, k integral.

Taking $\theta = \frac{2\pi}{9}k$, $k = 0, 1, K, 8$ gives 9 distinct numbers z with argument $\frac{2\pi}{9}k$.

If $z^6 + z^3 + 1 = 0$, then $z^9 = 1$ but $z^3 \neq 1$. Hence the roots of $z^6 + z^3 + 1 = 0$ are $\cos\left(\frac{2\pi}{9}k\right) + i\sin\left(\frac{2\pi}{9}k\right)$, $k = 1, 2, 4, 5, 7, 8$.

$\therefore\ z^6 + z^3 + 1 = 0$ has the roots $\operatorname{cis}\left(\pm\frac{2\pi}{9}\right)$, $\operatorname{cis}\left(\pm\frac{4\pi}{9}\right)$, $\operatorname{cis}\left(\pm\frac{8\pi}{9}\right)$.

13 *Solution*



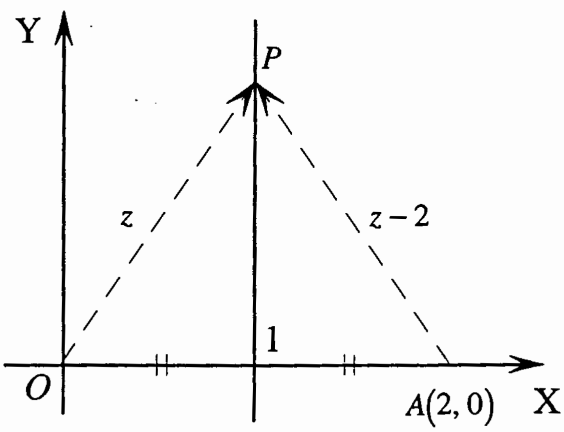
$|z|=6$ is the circle, center $(0,0)$ and radius 6. $|z-5|=5$ is the circle, center $(5,0)$ and the radius 5. Since y-axis is a tangent line to the circle $|z-5|=5$ at point $(0,0)$, if $|z|\le 6$ and $|z-5|\le 5$, then $-\frac{\pi}{2}<\arg z<\frac{\pi}{2}$.
Let $z=x+iy$. Then $|z|=6 \Rightarrow x^2+y^2=36$, and $|z-5|=5 \Rightarrow (x-5)^2+y^2=2$. Hence, if z such that both $|z|=6$ and $|z-5|=5$, then both $x^2+y^2=36$ and $x^2+y^2-10x+25=25$. Therefore $10x=36$.

$\therefore \quad x=\dfrac{18}{5}$.

$\therefore \quad y=\pm\sqrt{36-\left(\frac{18}{5}\right)^2}=\dfrac{24}{5}$.

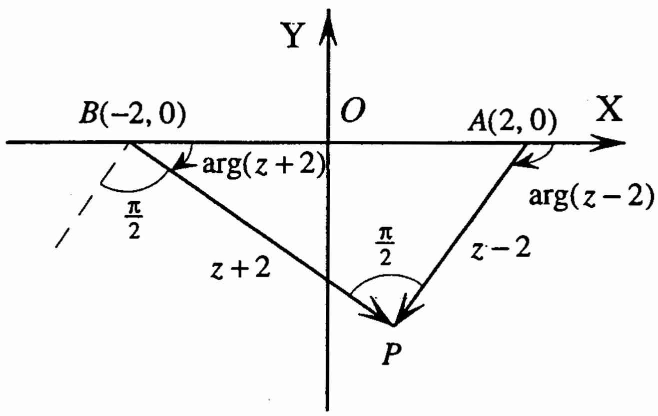
Hence the values of z for which both $|z|=6$ and $|z-5|=5$ are $\frac{18}{5}\pm i\frac{24}{5}$.

14 *Solution*

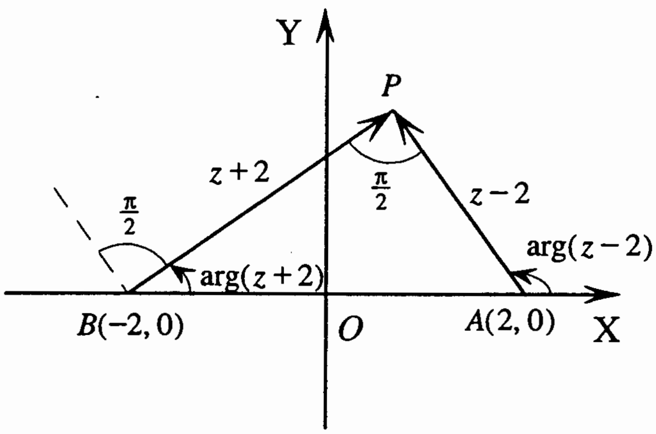


(a) Let A represent 2. Then $\overrightarrow{AP}$ represents $z-2$, and $|z|=|z-2| \Rightarrow OP=AP$. The locus of P is the perpendicular bisector of OA. Therefore the locus of P has Cartesian equation $x=1$.

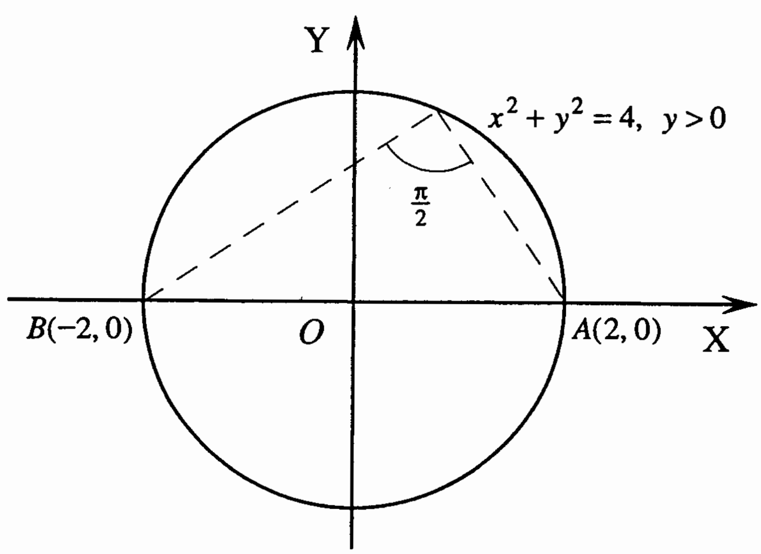
(b) Let $A(2,0)$, $B(-2,0)$ represent 2, -2 respectively. Then $\overrightarrow{AP}$ and $\overrightarrow{BP}$ represent $z-2$ and $z+2$ respectively. $\arg(z-2)=\arg(z+2)+\frac{\pi}{2}$ requires $\overrightarrow{AP}$ to be parallel to the vector obtained by rotation of $\overrightarrow{BP}$ anticlockwise through an angle of $\frac{\pi}{2}$.



If P lies below the x-axis, AP must be parallel to a clockwise rotation of BP. This diagram shows $\arg(z-2)=\arg(z+2)-\frac{\pi}{2}$. Hence P must lie above the x-axis.



Since alternate angles between parallel lines are equal, $\angle BPA=\frac{\pi}{2}$ as P traces its locus. Hence P lies on the upper arc AB of a circle through A and B.



The centre of this circle is the centre of diameter AB. Hence the locus of P has equation $x^2+y^2=4,\ \ y>0$, or $y=\sqrt{4-x^2}$.

Let $z=x+iy$ satisfies both $|z|=|z-2|$ and $\arg(z-2)=\arg(z+2)+\frac{\pi}{2}$. Then $x=1$ and $y=\sqrt{4-1}=\sqrt{3}$. Hence $z=1+i\sqrt{3}$.