7SD Solutions Series

Worked Solutions to Popular Mathematics Texts

Suggested Worked Solutions to

"4 Unit Mathematics"

(Text book for the NSW HSC by D. Arnold and G. Arnold)

Chapter 7
Mechanics

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Solutions are to "4 Unit Mathematics"

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Exercise 7.1

1 Solution

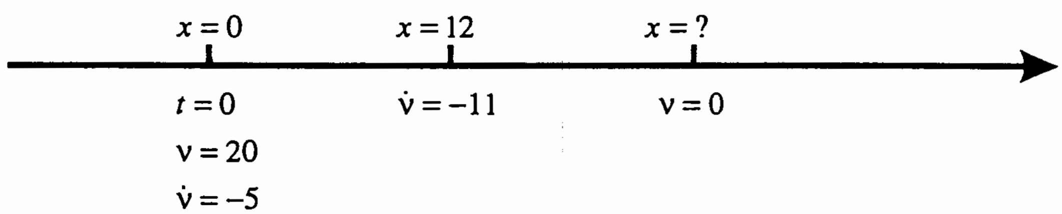
Expression relating $\ddot{x} = \dot{v}$ and v: $\dot{v} = \frac{k}{t^3}$, k constant. This equation has solution

$v = C - \frac{k}{2t^2}$, C constant. $\quad t \to +\infty,\ v \to 5\text{ms}^{-1} \Rightarrow C = 5 \Rightarrow v = 5 - \frac{k}{2t^2}$.

$t = 1,\ v = 3 \Rightarrow k = 4 \Rightarrow v = 5 - \frac{2}{t^2}$.

2 Solution

Choose initial position as origin, and initial direction as positive. When a particle is in rest, its velocity equals zero. So we can draw a figure with conditions of motion as follows



Equation of motion: $\ddot{x} = C - kx$, $C, k > 0$ constants, i. e.

$\dot{v} = C - kx$,

$x = 0, \dot{v} = -5 \Rightarrow C = -5 \Rightarrow \dot{v} = -5 - kx$,

$x = 12, \dot{v} = -11 \Rightarrow k = 1/2 \Rightarrow \dot{v} = -5 - \frac{x}{2}$.

So we obtained the expression relating x and v

$\dot{v} = -5 - \frac{x}{2}, \quad \frac{1}{2}\frac{dv^2}{dx} = -5 - \frac{x}{2}, \quad \frac{v^2}{2} + A = -5x - \frac{x^2}{4}$, A constant,

$x = 0,\ v = 20 \Rightarrow A = -200 \Rightarrow \frac{v^2}{2} = 200 - 5x - \frac{x^2}{4}$,

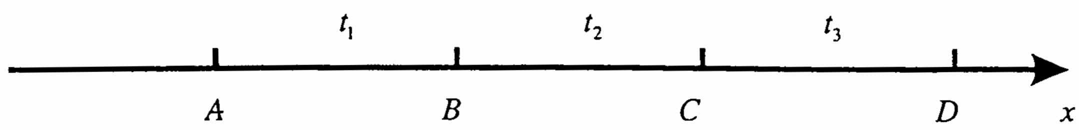
$v = 0 \Rightarrow x^2 + 20x - 800 = 0 \Rightarrow x = 20$.

Hence the particle moved a distance of $20m$ before coming to rest.

3 Solution

(a) $v = \frac{k}{x} \Rightarrow \ddot{x} = v\frac{dv}{dx} = \frac{k}{x}\left(\frac{-k}{x^2}\right) = \frac{-k^2}{x^3}$, i. e., $\ddot{x} = \frac{-k^2}{x^3}$.

(b) Let t_1, t_2 and t_3 be the times taken to travel the distances AB, BC and CD respectively.



$$B-A=C-B=D-C \Rightarrow B=\frac{A+C}{2},\ C=\frac{B+D}{2} \text{ and } C-A=D-B;$$

$$v=\frac{k}{x} \Rightarrow \frac{dx}{dt}=\frac{k}{x} \Rightarrow xdx=kdt.$$

By integrating the last equation $t_1=\frac{B^2-A^2}{2k}$, $t_2=\frac{C^2-B^2}{2k}$ and $t_3=\frac{D^2-C^2}{2k}$.

The numbers t_1, t_2 and t_3 are in arithmetic progression if and only if $t_2-t_1=t_3-t_2$.
But

$$t_2-t_1=\frac{A^2+C^2-2B^2}{2k}=\frac{A^2+C^2-2\left(\frac{A+C}{2}\right)^2}{2k}=\frac{A^2+C^2-2AC}{4k}=\frac{(A-C)^2}{4k},$$

analogously $t_3-t_2=\frac{(D-B)^2}{4k}$. Hence $t_2-t_1=t_3-t_2 \Leftrightarrow A-C=D-B$. is true.

4 Solution

Choose initial position as the origin and initial direction as positive.

Equation of motion: $\dot{v}=-kv^2$, $k>0$ constant.

(a) $\dot{v}=-kv^2 \Rightarrow \frac{1}{2}\frac{dv^2}{dx}=-kv^2 \Rightarrow \frac{dv^2}{v^2}=-2k\,dx \Rightarrow \ln v^2+C=-2kx$, C constant,

$x=0, v=V \Rightarrow C=-\ln V^2 \Rightarrow \ln\left(\frac{v^2}{V^2}\right)=-2kx \Rightarrow v=Ve^{-kx}$.

This is the required expression.

(b) $\dot{v}=-kv^2 \Rightarrow \frac{dv}{dt}=-kv^2 \Rightarrow \frac{dv}{v^2}=-k\,dt \Rightarrow \frac{-1}{v}+C=-kt$, C constant.

$t=0, v=V \Rightarrow C=\frac{1}{V} \Rightarrow \frac{1}{v}-\frac{1}{V}=kt \Rightarrow v=\frac{V}{1+Vkt}$. And we found the expression $v=v(t)$.

Further, from (a) $v=Ve^{-kx} \Rightarrow \frac{dx}{dt}=Ve^{-kx} \Rightarrow e^{kx}dx=Vdt \Rightarrow \frac{e^{kx}}{k}+C=Vt$, C constant.

$t=0, x=0 \Rightarrow C=-\frac{1}{k} \Rightarrow \frac{e^{kx}}{k}-\frac{1}{k}=Vt \Rightarrow x=\frac{1}{k}\ln\{Vkt+1\}$.

5 Solution

(a) $\frac{1}{v}=A+Bt \Rightarrow \frac{d}{dt}\left(\frac{1}{v}\right)=B \Rightarrow \frac{-\dot{v}}{v^2}=B \Rightarrow \dot{v}=-Bv^2$.

(b) $\frac{1}{v}=A+Bt$, $t=0,\ v=80 \Rightarrow A=\frac{1}{80}$. From (a) $\dot{v}=-Bv^2$;

$t=0,\ v=80,\ \dot{v}=-1 \Rightarrow B=\frac{1}{6400}$.

Furthermore,

$$v=\frac{1}{A+Bt} \Rightarrow \frac{dx}{dt}=\frac{1}{A+Bt} \Rightarrow dx=\frac{dt}{A+Bt} \Rightarrow x+C=\frac{\ln(A+Bt)}{B},\ C \text{ constant.}$$

$t=0,\ x=0 \Rightarrow C=\frac{\ln A}{B} \Rightarrow x=\frac{\ln\left(1+\frac{B}{A}t\right)}{B}$. Substituting the values of $A=\frac{1}{80}$ and

$B=\frac{1}{6400}$,

$$x=6400\ln\left(1+\frac{t}{80}\right).$$

At the same time from (a)

$$\dot{v}=-Bv^2 \Rightarrow \frac{1}{2}\frac{dv^2}{dx}=-Bv^2 \Rightarrow \frac{dv^2}{v^2}=-2B\,dx \Rightarrow \ln v^2+C=-2Bx.$$

$x=0,\ v=80 \Rightarrow C=-\ln 6400 \Rightarrow \ln\frac{v^2}{6400}=-2Bx \Rightarrow v=80e^{-Bx}$. Substituting

$B=\frac{1}{6400}$, $v=80\cdot e^{-x/6400}$.

6 Solution

Equation of motion: $m\ddot{x}=-kmv^3 \Rightarrow \dot{v}=-kv^3$, $k>0$ constant.

Relation between v and t: $\dot{v}=-kv^3 \Rightarrow \frac{dv}{dt}=-kv^3 \Rightarrow \frac{dv}{v^3}=-k\,dt \Rightarrow \frac{-1}{2v^2}+C=-kt$,

C constant, and

$$t=0, v=u \Rightarrow C=\frac{1}{2u^2} \Rightarrow \frac{1}{2v^2}=\frac{1}{2u^2}+kt \Rightarrow v=\frac{u}{\sqrt{1+2ku^2t}}.$$

Relation between v and x:

$$\dot{v}=-kv^3 \Rightarrow v\frac{dv}{dx}=-kv^3 \Rightarrow \frac{dv}{v^2}=-k\,dx \Rightarrow \frac{-1}{v}+C=-kx,$$

C constant.

$$x=0, v=u \Rightarrow C=\frac{1}{u} \Rightarrow \frac{1}{v}=\frac{1}{u}+kx \Rightarrow v=\frac{u}{1+kux}.$$

7 Solution

Choose initial position as the origin and initial direction as positive.

Equation of motion: $m\ddot{x}=-mk-mv^2 \Rightarrow \dot{v}=-k-v^2$, $k>0$ constant.

Initial conditions: $t=0,\ x=0,\ v=u$.

The distance travelled as the particle is brought to rest:

$$\dot{\mathrm{v}} = -k - \mathrm{v}^2 \Rightarrow \frac{1}{2}\frac{d\,\mathrm{v}^2}{dx} = -k - \mathrm{v}^2 \Rightarrow$$

$$\frac{d\,\mathrm{v}^2}{k+\mathrm{v}^2} = -2dx \Rightarrow \ln\left(k+\mathrm{v}^2\right) + C = -2x,\ C \text{ constant, and}$$

$$x = 0,\ \mathrm{v} = u \Rightarrow C = -\ln\left(k+u^2\right) \Rightarrow \ln\left(\frac{k+\mathrm{v}^2}{k+u^2}\right) = -2x.$$

When the particle is brought to rest its velocity is zero. Hence substituting $\mathrm{v} = 0$ into the last expression we obtain the travelled distance $x = \frac{1}{2}\ln\left(1+\frac{u^2}{k}\right)$.

The time taken as the particle is brought to rest: $\dot{\mathrm{v}} = -k - \mathrm{v}^2 \Rightarrow \frac{d\,\mathrm{v}}{k+\mathrm{v}^2} = -dt \Rightarrow$

$$\frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{\mathrm{v}}{\sqrt{k}}\right) + C = -t, \text{ and}$$

$$t = 0,\ \mathrm{v} = u \Rightarrow C = -\frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right) \Rightarrow t = \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right) - \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{\mathrm{v}}{\sqrt{k}}\right).$$

Substituting $\mathrm{v} = 0,\ t = \frac{1}{\sqrt{k}}\tan^{-1}\left(\frac{u}{\sqrt{k}}\right)$.

8 Solution

Choose initial position as the origin and initial direction as positive.

Equation of motion: $m\ddot{x} = -mk(\mathrm{v}^2 + C^2) \Rightarrow \dot{\mathrm{v}} = -k(\mathrm{v}^2 + C^2),\ k > 0$ constant.

(a) $x_{1/2}$ and $t_{1/2}$ denote the required distance and the required time respectively.

Relation between v and x :

$$\dot{\mathrm{v}} = -k(\mathrm{v}^2 + C^2) \Rightarrow \frac{1}{2}\frac{d\mathrm{v}^2}{dx^2} = -k(\mathrm{v}^2 + C^2) \Rightarrow \frac{d\mathrm{v}^2}{\mathrm{v}^2 + C^2} = -2kdx \Rightarrow$$

$$\ln\left(\mathrm{v}^2 + C^2\right) + A = -2kx,\ A \text{ constant.}$$

$$x = 0,\ \gamma = 2C \Rightarrow A = -\ln\left(5C^2\right) \Rightarrow x = \frac{1}{2k}\ln\left(\frac{5C^2}{\mathrm{v}^2 + C^2}\right) \tag{1}$$

The halved speed $\mathrm{v} = C \Rightarrow x_{1/2} = \frac{1}{2k}\ln\left(\frac{5}{2}\right)$.

Relation between v and t :

$$\dot{\mathrm{v}} = -k(\mathrm{v}^2 + C^2) \Rightarrow \frac{d\mathrm{v}}{\mathrm{v}^2 + C^2} = -kdt \Rightarrow \frac{1}{C}\tan^{-1}\frac{\mathrm{v}}{C} + A = -kt,$$

A constant, and $t = 0, \mathrm{v} = 2C \Rightarrow A = -\frac{1}{C}\tan^{-1}2 \Rightarrow$

$$t = \frac{1}{kC}\left(\tan^{-1}2 - \tan^{-1}\frac{\mathrm{v}}{C}\right). \tag{2}$$

If $\nu = C \Rightarrow t_{1/2} = \frac{1}{kC}\left(\tan^{-1}2 - \tan^{-1}1\right)$, but $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\frac{1}{3}$. Hence

$t_{1/2} = \frac{1}{kC}\tan^{-1}\frac{1}{3}$.

(b) Denote x_0 and t_0 the distance and the time for the particle to come to rest. Then from (1), if $\nu = 0, \Rightarrow x_0 = \frac{1}{2k}\ln 5$.

And hence the additional distance $x_0 - x_{1/2} = \frac{1}{2k}\left(\ln 5 - \ln\frac{5}{2}\right) = \frac{1}{2k}\ln 2$.

From (2), if $\nu = 0, \Rightarrow t_0 = \frac{1}{kC}\tan^{-1}2$.

Hence the additional time $t_0 - t_{1/2} = \frac{1}{kC}\left(\tan^{-1}2 - \tan^{-1}\frac{1}{3}\right)$.

But $\tan^{-1}2 - \tan^{-1}\frac{1}{3} = \tan^{-1}1 = \frac{\pi}{4} \Rightarrow t_0 - t_{1/2} = \frac{1}{kC}\cdot\frac{\pi}{4}$.

9 Solution

Equation of motion: $m\ddot{x} = -\frac{mgr^2}{x} \Rightarrow \dot{\nu} = -\frac{gr^2}{x^2}$.

Initial conditions: $x = r$, $\nu = u$.

(a) $\dot{\nu} = -\frac{gr^2}{x^2} \Rightarrow \frac{1}{2}\frac{d\nu^2}{dx} = -\frac{gr^2}{x} \Rightarrow d\nu^2 = -2gr^2\frac{dx}{x^2} \Rightarrow \nu^2 + C = \frac{2gr^2}{x}$, C constant.

$x = r, \nu = u \Rightarrow C = 2gr - u^2 \Rightarrow \nu^2 = u - 2gr + \frac{2gr^2}{x} \Rightarrow$

$$\nu = \sqrt{u^2 - 2gr\left(1 - \frac{r}{x}\right)}. \qquad (1)$$

(b) The particle will escape from the attraction of the earth if $\nu \to \nu_\infty > 0$ as $x \to +\infty$. But from (1) $\nu_\infty = \sqrt{u^2 - 2gr}$ and hence $\nu_\infty > 0$ as $u^2 > 2gr$.

10 Solution

$T = 2\pi\sqrt{\frac{l}{9{,}81}}$ is the period of the pendulum at ground level and $\tilde{T} = 2\pi\sqrt{\frac{l}{9{,}8}}$ is the period at mountain level. Then $\frac{\tilde{T}}{T} = \sqrt{\frac{9{,}81}{9{,}8}} \Rightarrow \tilde{T} = T\sqrt{\frac{9{,}81}{9{,}8}} \Rightarrow \tilde{T} - T = T\left(\sqrt{\frac{9{,}81}{9{,}8}} - 1\right)$.

The pendulum will be wrong per T seconds by $\tilde{T} - T$ seconds. Hence it will be wrong per every second by $\frac{\tilde{T} - T}{T}$ seconds. There are $24 \cdot 3600$ seconds in a day, therefore the pendulum will be wrong per day by

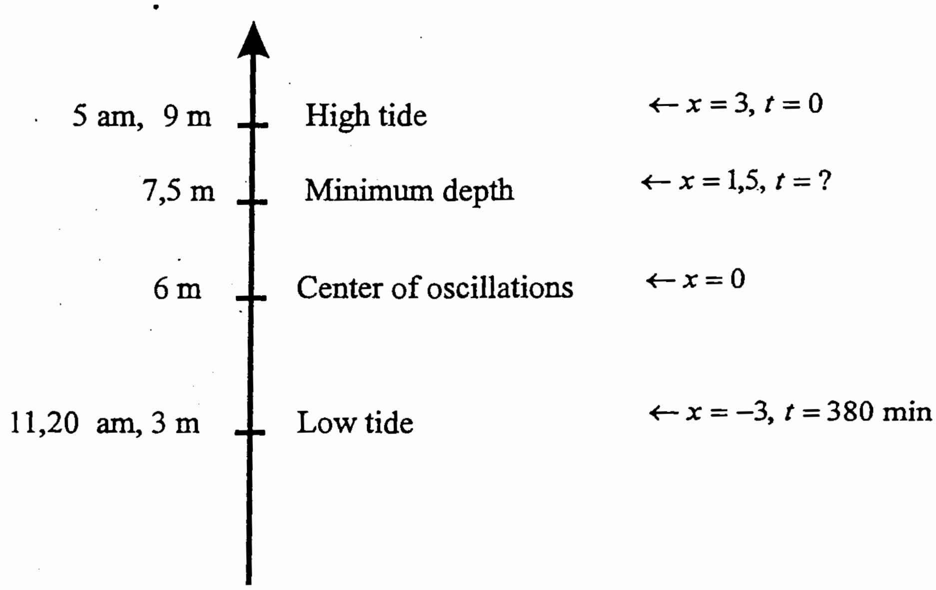
$24 \cdot 3600 \cdot \frac{\left(\tilde{T} - T\right)}{T} = 24 \cdot 3600\left(\sqrt{\frac{9{,}81}{9{,}8}} - 1\right) = 44$ s.

11 Solution

Let $T = 2\pi\sqrt{\dfrac{l}{9{,}81}}$ and $\tilde{T} = 2\pi\sqrt{\dfrac{l}{g}}$ be periods of the pendulum at ground level and at the new location respectively. Then the pendulum will be wrong per every second by $\dfrac{\tilde{T}-T}{T} = \sqrt{\dfrac{9{,}81}{g}} - 1$ seconds. Hence it will be wrong per day by $24\cdot 3600\cdot\left(\dfrac{\tilde{T}-T}{T}\right)$ seconds. So we obtain the following equation $\left(\sqrt{\dfrac{9{,}81}{9{,}8}}-1\right)24\cdot 3600 = 30$,

$$g = \frac{9{,}81}{\left(1+\dfrac{30}{24\cdot 3600}\right)^2}, \qquad g = 9{,}803\, ms^{-2}.$$

12 Solution



Period $T = 2\cdot(11{,}20-5) = 2\cdot 380 = 760$ minutes

Amplitude is $\frac{1}{2}(9-3) = 3m$.

Motion is simple harmonic $\Rightarrow \ddot{x} = -n^2x,\ n = \dfrac{2\pi}{T} = \dfrac{\pi}{380}$.

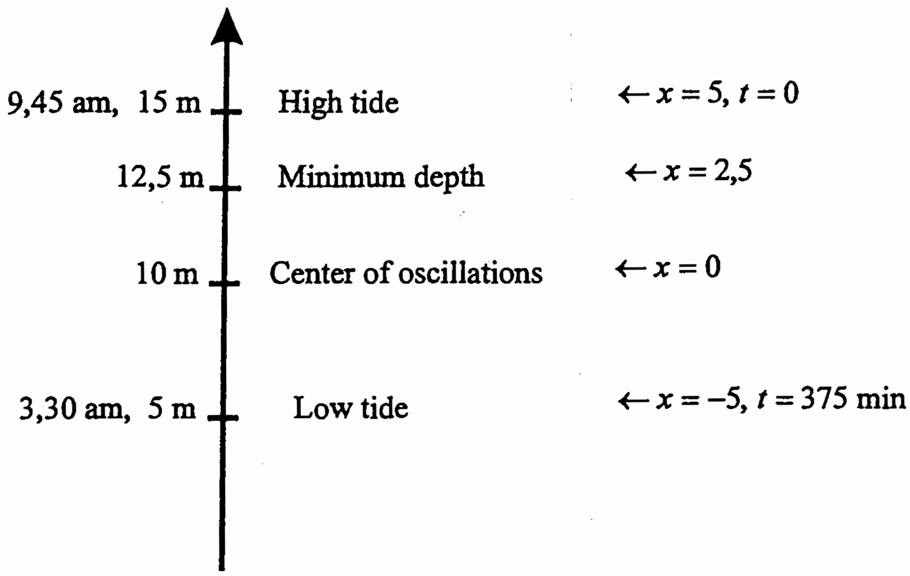
This equation has solution $x = 3\cos(nt+\alpha),\ 0 \le \alpha < 2\pi$.

Initial conditions: $t = 0,\ x = 3 \Rightarrow \cos\alpha = 1 \Rightarrow \alpha = 0 \Rightarrow x = 3\cos nt$. A minimum depth is $7{,}5m$ if $x = 1{,}5 \Rightarrow$ we have the equation

$$1{,}5 = 3\cos nt \Rightarrow \cos nt = \frac{1}{2} \Rightarrow nt = \frac{\pi}{3} \Rightarrow t = \frac{\pi}{3n} = \frac{380}{3} = 2{,}06.$$

Hence the latest time before noon when a minimum depth of $7{,}5m$ of water is $5+2{,}06=7{,}06$.

13 Solution



Period $T=2\cdot 375=750$ minutes.

Amplitude is $\frac{1}{2}(15-5)=5m$.

Motion is simple harmonic $\Rightarrow \ddot{x}=-n^2x,\ n=\frac{2\pi}{T}=\frac{\pi}{375}$.

This equation has solution $x=5\cos(nt+\alpha),\ 0\le\alpha<2\pi$.

Initial conditions: $t=0,\ x=5\Rightarrow\cos\alpha=1\Rightarrow\alpha=0\Rightarrow x=5\cos nt$.

(a) The level of water of $12{,}5m$ corresponds to $x=2{,}5\Rightarrow$ we have the equation

$2{,}5=5\cos nt,\quad \cos nt=\frac{1}{2},\ nt=-\frac{\pi}{3}$ or $nt=\frac{\pi}{3},\ t=-\frac{375}{3}$ or $t=\frac{375}{3}$,

i. e., $t=-2{,}05$ or $t=2{,}05$.

As $t=0$ corresponds to $9{,}45\ am$, the ship can safely enter the harbour between

$t_1=9{,}45-2{,}05=7{,}40\ am$ and $t_2=9{,}45+2{,}05=11{,}50\ am$

(b) The level of water of $13\ m$ corresponds to $x=3$. Hence

$3=5\cos nt\Rightarrow nt=\cos^{-1}\frac{3}{5}$.Then

$$|v|=|\dot{x}|=5n\sin nt=5n\sin\cos^{-1}\frac{3}{5}\Rightarrow|v|=5\cdot\frac{\pi}{375}\sqrt{1-\left(\frac{3}{5}\right)^2}=0{,}034m\ \min^{-1}.$$

Exercise 7.2

1 Solution

Choose point 0 as the origin and $\downarrow$ as the positive direction.

Equation of motion: $\dot{v} = g - kv$.

Initial conditions: $t = 0,\ x = 0,\ v = 0$.

Terminal velocity V hence $g = kV \Rightarrow k = g / V$.

Expression relating x and v

$$\dot{v} = g - kv,$$

$$v\frac{dv}{dx} = g - kv,$$

$$-kdx = \frac{-kvdv}{g - kv},$$

$$-kdx = \left\{1 - \frac{g}{g - kv}\right\}dv,$$

$$-k^2dx = kdv + g\cdot\frac{-kdv}{g - kv},$$

constant,

$$-k^2x + C = kv + g\ln|g - kv| \quad C \text{ constant},$$

$$x = 0,\ v = 0 \Rightarrow C = g\ln g,$$

$$x = -\frac{v}{k} + \frac{g}{k^2}\ln\left|\frac{g}{g - kv}\right|. \qquad (1)$$

Expression relating v and t

$$\dot{v} = g - kv,$$

$$\frac{dv}{dt} = g - kv,$$

$$dt = \frac{dv}{g - kv},$$

$$-kdt = \frac{-kdv}{g - kv},$$

$$-kt + C = \ln|g - kv| \quad C$$

$$t = 0,\ v = 0 \Rightarrow C = \ln g,$$

$$t = \frac{1}{k}\ln\left|\frac{g}{g - kv}\right|. \qquad (2)$$

From (2) $\ln\left|\dfrac{g}{g - kv}\right| = kt \Rightarrow$ from (1) $x = -\dfrac{v}{k} + \dfrac{g}{k}t$. But $k = g / V$,

hence $x = -\dfrac{vV}{g} + Vt \Rightarrow\ xg + Vv = Vgt$.

2 Solution

Choose point 0 as the origin and $\downarrow$ as the positive direction.

Equation of motion: $\dot{v} = g - kv^2$.

Initial conditions: $t=0,\ x=0,\ \text{v}=0$.

Terminal velocity V hence $g = kV^2 \Rightarrow k = g/V^2$.

Relation between v and x:

$$\dot{\text{v}} = g - k\text{v}^2,$$

$$\frac{1}{2}\frac{d\text{v}^2}{dx} = g - k\text{v}^2,$$

$$-2k\,dx = \frac{-k\,d\text{v}^2}{g-k\text{v}^2},$$

$$-2kx + C = \ln\left|g-k\text{v}^2\right| \quad C \text{ constant},$$

$$x=0,\ \text{v}=0 \Rightarrow C = \ln g,$$

$$x = \frac{1}{2k}\ln\left|\frac{g}{g-k\text{v}^2}\right|. \qquad (1)$$

Relation between v and t:

$$\dot{\text{v}} = g - k\text{v}^2,$$

$$\frac{d\text{v}}{dt} = g - k\text{v}^2,$$

$$\sqrt{k}\,dt = \frac{\sqrt{k}\,d\text{v}}{g-\left(\sqrt{k}\,\text{v}\right)^2},$$

$$\sqrt{k}\,dt = \left\{\frac{\sqrt{k}}{\sqrt{g}-\sqrt{k}\,\text{v}} + \frac{\sqrt{k}}{\sqrt{g}+\sqrt{k}\,\text{v}}\right\}\frac{d\text{v}}{2\sqrt{g}},$$

$$2\sqrt{kg}\,t + C = \ln\left|\frac{\sqrt{g}+\sqrt{k}\,\text{v}}{\sqrt{g}-\sqrt{k}\,\text{v}}\right|,$$

$$t=0,\ \text{v}=0 \Rightarrow C=0,$$

$$t = \frac{1}{2\sqrt{kg}}\ln\left|\frac{\sqrt{g}+\sqrt{k}\text{v}}{\sqrt{g}-\sqrt{k}\text{v}}\right|. \ (2)$$

If $\text{v} = \dfrac{V}{2}$ and $k = g/V^2$, then from (1) $x = \dfrac{V^2}{2g}\ln\dfrac{4}{3}$ and from (2) $t = \dfrac{V}{2g}\ln 3$.

3 Solution

1. Free motion: choose initial point as the origin and $\downarrow$ as the positive direction.

Equation of motion: $\dot{\text{v}} = g$.

Initial conditions: $t=0,\ x=0,\ \text{v}=0$.

Find $\text{v}=\text{v}(t)$ and $x = x(t)$:

$\dot{\text{v}} = g \Rightarrow \text{v} + C = gt,\ C$ constant;

$t=0,\ \text{v}=0 \Rightarrow C=0 \Rightarrow \text{v}=gt \Rightarrow \dot{x}=gt \Rightarrow x + C = \dfrac{gt^2}{2},\ C$ constant;

$t=0,\ x=0 \Rightarrow x=\dfrac{gt^2}{2}$.

Let v and x at a time $t=\dfrac{1}{2k}$ be V and h respectively. Then from the relations $\mathrm{v}=gt$ and $x=\dfrac{gt^2}{2}$ we obtain $V=\dfrac{g}{2k}$ and $h=\dfrac{g}{8k^2}$.

2. Motion with resistance: choose the point where the parachutist opened his parachute as the origin and $\downarrow$ as the positive direction.

Equation of motion: $\dot{\mathrm{v}}=g-k\mathrm{v}$.

Initial conditions: $t=0,\ x=0,\ \mathrm{v}=V$.

Relation between x and v:

$$\dot{\mathrm{v}}=g-k\mathrm{v} \Rightarrow \mathrm{v}\frac{d\mathrm{v}}{dx}=g-k\mathrm{v} \Rightarrow dx=\frac{\mathrm{v}\,d\mathrm{v}}{g-k\mathrm{v}} \Rightarrow -kdx=\frac{-k\mathrm{v}\,d\mathrm{v}}{g-k\mathrm{v}} \Rightarrow$$

$$-kdx=\left(1+\frac{-g}{g-k\mathrm{v}}\right)d\mathrm{v} \Rightarrow -k^2dx=\left(k+g\frac{-k}{g-k\mathrm{v}}\right)d\mathrm{v} \Rightarrow -k^2x+C=k\mathrm{v}+g\ln\left|g-k\mathrm{v}\right|,\ C$$

constant;

$$x=0,\ \mathrm{v}=V \Rightarrow C=kV+g\ln\left|g-kV\right| \Rightarrow x=\frac{g}{k^2}\ln\left|\frac{g-kV}{g-k\mathrm{v}}\right|-\frac{(\mathrm{v}-V)}{k}.$$

But $V=\dfrac{g}{2k}$ and if $\mathrm{v}=\dfrac{3g}{4k}$, then $x=\dfrac{g}{k^2}\ln 2-\dfrac{g}{4k^2}$.

The total distance the parachutist has fallen is $h+x$:

$$h+x=\frac{g}{k^2}\ln 2-\frac{g}{4k^2}+\frac{g}{8k^2},$$

$$h+x=\frac{g}{8k^2}(8\ln 2-1).$$

4 Solution

Choose initial position as the origin and $\downarrow$ as positive direction.

Equation of motion: $\dot{\mathrm{v}}=g-k\mathrm{v}^2$.

Initial conditions: $t=0,\ x=0,\ \mathrm{v}=0$.

(a) Find relation between x and v:

$$\dot{v} = g - kv^2 \Rightarrow \frac{1}{2}\frac{dv^2}{dx} = g - kv^2 \Rightarrow 2\,dx = \frac{dv^2}{g - kv^2} \Rightarrow -2k\,dx = \frac{-k\,dv^2}{g - kv^2} \Rightarrow$$

$$-2kx + C = \ln\left|g - kv^2\right|;$$

$$x = 0,\ v = 0 \Rightarrow C = \ln g \Rightarrow -2kx = \ln\left|1 - \frac{k}{g}v^2\right| \Rightarrow$$

$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right). \qquad (1)$$

(b) From (1) $v_1^2 = \frac{g}{k}\left(1 - e^{-2kd_1}\right) \Rightarrow e^{-2kd_1} = 1 - v_1^2\frac{k}{g} \Rightarrow$

$$e^{-2k(2d_1)} = \left(1 - v_1^2\frac{k}{g}\right)^2. \qquad (2)$$

From (1) $\left(\frac{5}{4}v_1\right)^2 = \frac{g}{k}\left(1 - e^{-2k(2d_1)}\right)$ and taking account of (2)

$$\left(\frac{5}{4}v_1\right)^2\frac{k}{g} = 1 - \left(1 - v_1^2\frac{k}{g}\right)^2 \Rightarrow \frac{25}{16}v_1^2\frac{k}{g} = 2v_1^2\frac{k}{g} - v_1^4\left(\frac{k}{g}\right)^2 \Rightarrow \frac{k}{g} = \frac{7}{16}\frac{1}{v_1^2}.$$

As the resistance to the particle's motion is mkv^2, the greatest possible speed of the particle is $v = \sqrt{\frac{g}{k}}$. But $\frac{k}{g} = \frac{7}{16}\cdot\frac{1}{v_1^2} \Rightarrow v = \frac{4v_1}{\sqrt{7}}$.

5 Solution

(a) Choose initial position as the origin and initial direction $\uparrow$ as positive.

Equation of motion: $\dot{v} = -g - kv$.

Initial conditions: $t = 0,\ x = 0,\ v = u$.

Relation between x and v:

$$v\frac{dv}{dx} = -(g + kv),$$

$$-dx = \frac{v\,dv}{g + kv},$$

$$-k\,dx = \frac{kv\,dv}{g + kv},$$

$$-k\,dx = \left(1 - \frac{g}{g + kv}\right)dv,$$

Relation between v and t:

$$\frac{dv}{dt} = -(g + kv),$$

$$-dt = \frac{dv}{g + kv},$$

$$-k\,dt = \frac{k\,dv}{g + kv},$$

$$-kt + C = \ln\left|g + kv\right|,$$

$$-k^2dx = \left(k - g\frac{k}{g+kv}\right)dv,$$

$$t=0,\ v=u \Rightarrow C = \ln|g+ku|,$$

$$-k^2x + C = kv - g\ln|g+kv|,$$

$$t = \frac{-1}{k}\ln\left|\frac{g+kv}{g+ku}\right|. (2)$$

$$x=0,\ v=u \Rightarrow C = ku - g\ln|g+ku|,$$

$$x = \frac{u-v}{k} + \frac{g}{k^2}\ln\left|\frac{g+kv}{g+ku}\right|. \qquad (1)$$

If the particle reaches its greatest height H at a time T, its speed v is equal to zero.

Hence from (1) $H = \frac{u}{k} + \frac{g}{k^2}\ln\left|\frac{g}{g+ku}\right|$. And from (2) $T = -\frac{1}{k}\ln\left|\frac{g}{g+ku}\right|$.

From here $\ln\left|\frac{g}{g+ku}\right| = -kT \Rightarrow H = \frac{u}{k} - \frac{gkT}{k^2} \Rightarrow u = kH + gT$.

(b) Choose the highest point reached as the origin and $\downarrow$ as the positive direction.

Equation of motion: $\dot{v} = g - kv$.

Initial conditions: $t=0,\ x=0,\ v=0$.

Expression relating x and v:

$$v\frac{dv}{dx} = g - kv,$$

$$dx = \frac{v\,dv}{g-kv},$$

$$-k\,dx = \frac{-kv\,dv}{g-kv},$$

$$-k\,dx = \left(1 - \frac{g}{g-kv}\right)dv,$$

$$-k^2dx = \left(k + g\cdot\frac{-k}{g-kv}\right)dv,$$

$$-k^2x + C = kv + g\ln|g-kv|,$$

$$x=0,\ v=0 \Rightarrow C = g\ln g,$$

Expression relating v and t:

$$\frac{dv}{dt} = g - kv,$$

$$dt = \frac{dv}{g-kv},$$

$$-kdt = \frac{-kdv}{g-kv},$$

$$-kt + C = \ln|g-kv|,$$

$$t=0,\ v=0 \Rightarrow C = \ln g,$$

$$t = \frac{1}{k}\ln\left|\frac{g}{g-kv}\right|. (4)$$

$$x=-\frac{\mathrm{v}}{k}+\frac{g}{k^2}\ln\left|\frac{g}{g-k\mathrm{v}}\right|. \qquad (3)$$

As the particle reaches its original position, $\mathrm{v}=w$ and $x=H$, hence from (3)

$H=-\frac{w}{k}+\frac{g}{k^2}\ln\left|\frac{g}{g-kw}\right|$. At the same time from (4) as $\mathrm{v}=w$ and $t=T'$ we obtain

$T'=\frac{1}{k}\ln\left|\frac{g}{g-kw}\right|$. From here $\ln\left|\frac{g}{g-kw}\right|=kT'\Rightarrow H=-\frac{w}{k}+\frac{g}{k^2}kT'\Rightarrow w=gT'-kH$.

6 Solution

(a) Upward motion.

Choose the point of projection as origin and $\uparrow$ as positive direction.

Equation of motion: $\dot{\mathrm{v}}=-g-k\mathrm{v}^2$.

Initial conditions: $t=0,\ x=0,\ \mathrm{v}=V$.

Expression relating x and v:

$$\frac{1}{2}\frac{d\mathrm{v}^2}{dx}=-(g+k\mathrm{v}^2)\Rightarrow -2\,dx=\frac{d\mathrm{v}^2}{g+k\mathrm{v}^2}\Rightarrow -2k\,dx=\frac{-k\,d\mathrm{v}^2}{g+k\mathrm{v}^2}\Rightarrow$$
$$-2kx+C=\ln\left|g+k\mathrm{v}^2\right|;$$

$x=0,\ \mathrm{v}=V\Rightarrow C=\ln\left|g+kV^2\right|\Rightarrow$

$$x=\frac{1}{2k}\ln\left|\frac{g+kV^2}{g+k\mathrm{v}^2}\right|. \qquad (1)$$

At the maximum height, $\mathrm{v}=0$. Let the maximum height be h. From (1)

$$h=\frac{1}{2k}\ln\left|1+\frac{k}{g}V^2\right|. \qquad (2)$$

(b) Downward motion.

Setting the origin to the maximum height attained and $\downarrow$ as the positive direction.

Equation of motion: $\dot{\mathrm{v}}=g-k\mathrm{v}$.

Initial conditions: $t=0,\ x=0,\ \mathrm{v}=0$.

Terminal velocity: as $\dot{\mathrm{v}}\to 0.\ \mathrm{v}\to\sqrt{\frac{g}{k}}=V$.

Expression relating x and v:

$$\frac{1}{2}\frac{dv^2}{dx} = g - kv^2 \Rightarrow 2\,dx = \frac{dv^2}{g - kv^2} \Rightarrow -2k\,dx = \frac{-k\,dv^2}{g - kv^2} \Rightarrow -2kx + C = \ln\left|g - kv^2\right|.$$

$$x = 0,\ v = 0 \Rightarrow C = \ln g \Rightarrow x = \frac{1}{2k}\ln\left|\frac{g}{g - k\,v^2}\right| \Rightarrow$$

$$x = \frac{1}{2k}\ln\left|\frac{1}{1 - \dfrac{k}{g}v^2}\right|. \tag{3}$$

As the terminal velocity $V = \sqrt{\dfrac{g}{k}}$, it follows from (2) that

$$h = \frac{1}{2k}\ln 2. \tag{4}$$

Let the speed of the particle when it returns to its projection point be u. Then from (3)

$$h = \frac{1}{2k}\ln\left|\frac{1}{1 - \dfrac{k}{g}u^2}\right|.$$

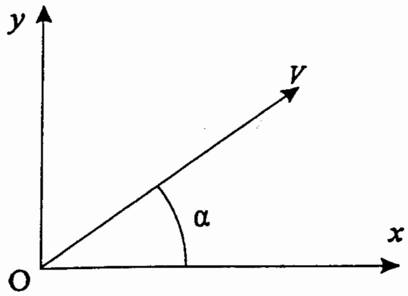
From here and from (4)

$$\frac{1}{1 - \left(k/g\right)u^2} = 2 \Rightarrow u^2 = \frac{g}{k}\cdot\frac{1}{2} \Rightarrow u = \sqrt{\frac{g}{k}}\cdot\frac{1}{\sqrt{2}} = \frac{V}{\sqrt{2}}.$$

Exercise 7.3

1 Solution

Axes and origin:



O is the point of projection

After t seconds the particle is at position:

$$x(t) = V\cos\alpha \cdot t, \tag{1}$$

$$y(t) = V\sin\alpha \cdot t - \frac{gt^2}{2}. \tag{2}$$

(a) As the particle hits the ground, its coordinates are $x = R$,

$y = 0$; $y = 0 \Rightarrow$ from (2) $t = \frac{2}{g}V\sin\alpha$;

$x = R \Rightarrow$ from (1) $R = V\cos\alpha \cdot t \Rightarrow R = V\cos\alpha \cdot \frac{2}{g}V\sin\alpha \Rightarrow \sin 2\alpha = \frac{gR}{V^2}$;

$R < \frac{V^2}{g} \Rightarrow \sin 2\alpha < 1 \Rightarrow 2\alpha = (-1)^k \arcsin\frac{gR}{V^2} + \pi k,\ k \in Z$.

But $0 < \alpha < \frac{\pi}{2} \Rightarrow 0 < 2\alpha < \pi$, hence $2\alpha = \arcsin\frac{gR}{V^2}$ or $2\alpha = \pi - \arcsin\frac{gR}{V^2}$.

So there are two possible angles of projection for a given R: $\alpha_1 = \frac{1}{2}\arcsin\frac{gR}{V^2}$ and

$\alpha_2 = \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{gR}{V^2}$. Obviously, $\alpha_1 + \alpha_2 = \frac{\pi}{2}$.

(b) From (1) $t = \frac{x}{V\cos\alpha}$, hence $t_1 = \frac{R}{V\cos\alpha_1}$ and $t_2 = \frac{R}{V\cos\alpha_2}$. So we have

$$t_1 \cdot t_2 = \frac{R^2}{V^2\cos\alpha_1\cos\alpha_2} = \frac{R^2}{V^2}\ \frac{2}{\cos(\alpha_2-\alpha_1)+\cos(\alpha_2+\alpha_1)};$$

$$t_1 \cdot t_2 = \frac{R^2}{V^2}\cdot\frac{2}{\cos\left(\frac{\pi}{2} - \arcsin\frac{gR}{V^2}\right)} = \frac{R^2}{V^2}\cdot\frac{2}{\frac{gR}{V^2}} = \frac{2}{g}R \quad \Rightarrow R = \frac{1}{2}g\,t_1 t_2.$$

As the particle attains its highest point, its velocity $\dot{y} = V\sin\alpha - gt$ is zero.

Furthermore, $\dot{y} = 0 \Rightarrow t = \dfrac{V\sin\alpha}{g}$ is the time when the particle reached its greatest height. Hence from (2) the greatest height is

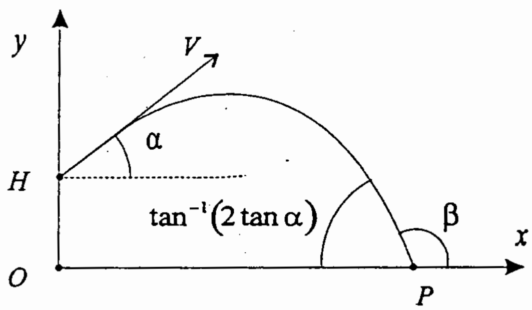
$$h = V\sin\alpha \cdot \frac{V\sin\alpha}{g} - \frac{g}{2}\frac{V^2}{g^2}\sin^2\alpha = \frac{V^2\sin^2\alpha}{2g}.$$

So we have $4\sqrt{h_1 h_2} = \dfrac{2V^2}{g}\sin\alpha_1 \cdot \sin\alpha_2 = \dfrac{V^2}{g}\left(\cos(\alpha_2 - \alpha_1) - \cos(\alpha_2 + \alpha_1)\right)$, and

$$4\sqrt{h_1 h_2} = \frac{V^2}{g}\frac{gR}{V^2} = R.$$

2 Solution

Axes, origin and path:



After t seconds the particle is at the position

$$x = V\cos\alpha \cdot t\ , \tag{1}$$

$$y = h + V\sin\alpha \cdot t - \frac{gt^2}{2}. \tag{2}$$

O is the foot of the cliff. $OH = h$. OP is the distance from the foot of the cliff to the point of impact.

(a) As the particle hits the ground, $y = 0$; $y = 0 \Rightarrow$ from (2) $\dfrac{gt^2}{2} - V\sin\alpha \cdot t - h = 0$.

Solving this quadratic, we obtain the time of flight

$$t = \frac{V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}}{g}. \tag{3}$$

Hence from (1) we get

$$OP = V\cos\alpha \cdot \left(\frac{V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}}{g}\right). \tag{4}$$

(b) Since $\tan\beta = y'_x = \dfrac{\dot{y}}{\dot{x}}$, but $\tan\beta = -2\tan\alpha \Rightarrow -2\tan\alpha = \dfrac{\dot{y}}{\dot{x}}$. Hence from (1) and (2) we get $-2\tan\alpha = \dfrac{V\sin\alpha - gt}{V\cos\alpha}$, and

$$t = \frac{3\sin\alpha}{g} \cdot V. \qquad (5)$$

Equating (5) and (3), we get

$$\frac{3\sin\alpha}{g} \cdot V = \frac{V\sin\alpha + \sqrt{V^2\sin^2\alpha + 2gh}}{g}; \quad 2\sin\alpha V = \sqrt{V^2\sin^2\alpha + 2gh};$$

$$V = \sqrt{\frac{2gh}{3}} \frac{1}{\sin\alpha}. \qquad (6)$$

Now we can express the distance OP in terms of h and α. Substituting (6) into (4), we obtain

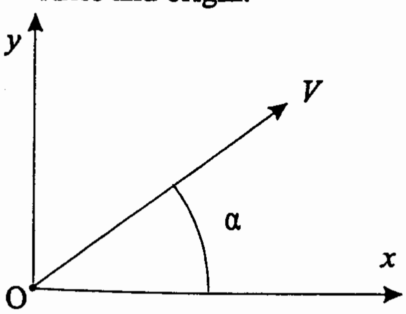
$$OP = \sqrt{\frac{2gh}{3}} \frac{\cos\alpha}{\sin\alpha} \left(\frac{\sqrt{\frac{2gh}{3}} + \sqrt{\frac{2gh}{3} + 2gh}}{g} \right); \qquad OP = 2h\cot\alpha.$$

3 Solution

After t seconds the particle is at the position:

$$x(t) = V\cos\alpha \cdot t, \qquad (1)$$

Axes and origin:



O is the point of prejection

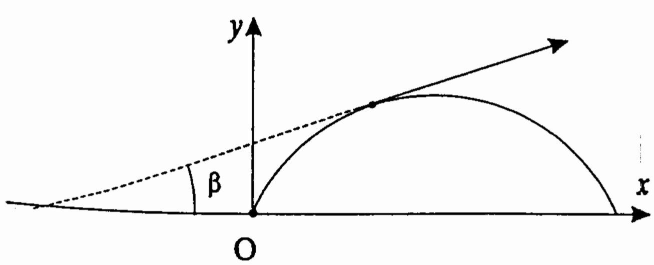
$$y(t) = V\sin\alpha \cdot t - \frac{gt^2}{2}. \qquad (2)$$

(a) $\tan\beta = y'_x = \frac{\dot{y}}{\dot{x}}$, hence from (1) and (2) we get

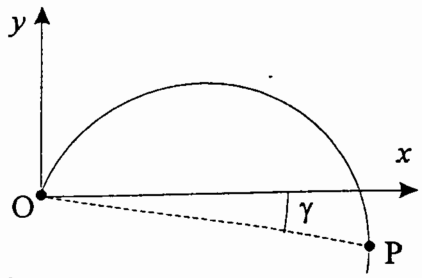
$\tan\beta = \frac{V\sin\alpha - gt}{V\cos\alpha}$, where

$t = \frac{T}{4} \Rightarrow \tan\beta = \tan\alpha - \frac{g}{V\cos\alpha}\frac{T}{4}$; $y = 0,\ t = T \Rightarrow$

from (2) $\frac{T}{4} = \frac{V\sin\alpha}{2g}$. And hence



$$\tan\beta = \tan\alpha - \frac{g}{V\cos\alpha} \cdot \frac{V\sin\alpha}{2g} = \frac{1}{2}\tan\alpha.$$



(b)

$$\tan\gamma = \frac{|y|}{x} = \frac{|V\sin\alpha\cdot t-(gt^2/2)|}{V\cos\alpha\cdot t}$$

$$\Rightarrow \frac{1}{3}\tan\alpha = \frac{gt^2/2 - V\sin\alpha\cdot t}{V\cos\alpha\cdot t} \Rightarrow$$

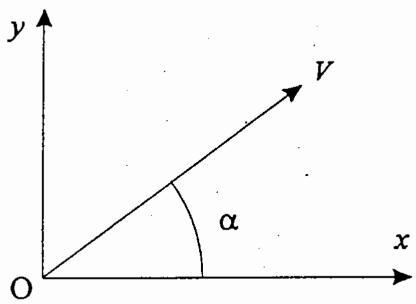
$$\frac{1}{3}\tan\alpha = \frac{gt}{2V\cos\alpha} - \tan\alpha$$

$$\Rightarrow t = \frac{4}{3}\cdot\left(\frac{2V\sin\alpha}{g}\right).\text{ But from (a) } T=\frac{2V\sin\alpha}{g} \Rightarrow t=\frac{4}{3}T.$$

4 Solution

After t seconds the particle is at the position:

Axes and origin:

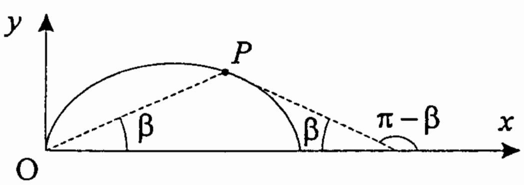


O is the point of prejection

$$x = V\cos\alpha\cdot t, \quad (1)$$

$$y = V\sin\alpha\cdot t - \frac{gt^2}{2}. \quad (2)$$

(a) Obviously $\tan\beta = \frac{y}{x}$, hence from (1) and (2)



$$\tan\beta = \frac{V\sin\alpha\cdot t - gt^2/2}{V\cos\alpha\cdot t} \Rightarrow$$

$$\tan\beta = \tan\alpha - \frac{g}{2V\cos\alpha}\cdot t \quad (3)$$

Also $\tan(\pi-\beta) = y'_x \Rightarrow$ from (1) and (2)

$$-\tan\beta = \frac{\dot{y}}{\dot{x}} = \frac{V\sin\alpha - gt}{V\cos\alpha};\ \tan\beta = \frac{g}{V\cos\alpha}t - \tan\alpha.$$

Equating this with (3), $\tan\alpha - \frac{g}{2V\cos\alpha}t = \frac{g}{V\cos\alpha}t - \tan\alpha;\ \frac{3}{2}\frac{g}{V\cos\alpha}t = 2\tan\alpha;$

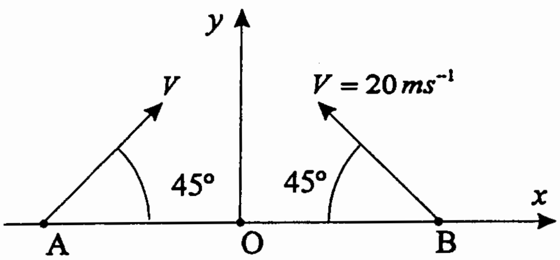
$$t = \frac{4}{3}\frac{V}{g}\sin\alpha \qquad (4)$$

(b) From (3) and (4) $\tan\beta = \tan\alpha - \frac{g}{2V\cos\alpha}\cdot\frac{4}{3}\frac{V}{g}\sin\alpha$;

$\tan\beta = \frac{1}{3}\tan\alpha \Rightarrow 3\tan\beta = \tan\alpha.$

5 Solution

Axes and origin:



O is the centre of AB, i. e. AO=OB= $20\,m$.

After t seconds the particle projected from A is at the position:

$$x = -\mathrm{AO} + V\cos\alpha\cdot t\,, \qquad (1)$$

$$y = V\sin\alpha\cdot t - gt^2/2\,. \qquad (2)$$

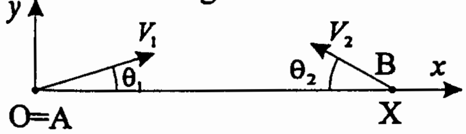
Because of the symmetry of conditions of the problem the particles collide over the point O.

$x = 0 \Rightarrow$ from (1) $t = \frac{\mathrm{AO}}{V\cos\alpha} \Rightarrow t = \frac{20}{20\cos 45^\circ} = \sqrt{2}\,s$;

$t = \sqrt{2} \Rightarrow$ from (2) $y = 20\cdot\sin 45^\circ\cdot\sqrt{2} - \frac{g2}{2} \Rightarrow y = (20 - g)m.$

6 Solution

Axes and origin:



Initial conditions: when $t = 0$

$x_1 = y_1 = 0,\qquad x_2 = -\mathrm{X},\ y_2 = 0,$

$\dot{x}_1 = V_1\cos\theta_1,\qquad \dot{x}_2 = -V_2\cos\theta_2,$

$\dot{y}_1 = V_1\sin\theta_1,\qquad \dot{y}_2 = V_2\sin\theta_2.$

Hence after t seconds, the two particles are at the positions:

$$x_1 = V_1\cos\theta_1\cdot t\,, \qquad (1) \qquad\qquad x_2 = \mathrm{X} - V_2\cos\theta_2\cdot t\,, \qquad (3)$$

$$y_1 = V_1 \sin\theta_1 \cdot t - gt^2/2, \qquad (2) \qquad\qquad y_2 = V_2 \sin\theta_2 \cdot t - gt^2/2. \qquad (4)$$

(a) The two particles collide at a time T if and only if at that moment their coordinates are equal. Hence

$t = T,\ x_1 = x_2$ and $t = T,\ y_1 = y_2$

from (1) and (3)$\Rightarrow$ from (2) and (4)$\Rightarrow$

$$V_1 \cos\theta_1 \cdot T = X - V_2 \cos\theta_2 \cdot T \qquad\qquad V_1 \sin\theta_1 = V_2 \sin\theta_2 \qquad (6)$$

$$X = (V_1 \cos\theta_1 + V_2 \cos\theta_2)\cdot T \qquad (5)$$

and this is the second condition.

(b)

$$\tan\theta_1 = \frac{4}{3} \Rightarrow \frac{\sin^2\theta_1}{\cos^2\theta_1} = \frac{16}{9} \Rightarrow \sin^2\theta_1 = (1-\sin^2\theta_1)\cdot\frac{16}{9} \Rightarrow$$

$$\sin\theta_1 = \frac{4}{5} \Rightarrow V_1 \sin\theta_1 = 45\cdot\frac{4}{5} = 36.$$

$$\tan\theta_2 = \frac{3}{4} \Rightarrow \sin\theta_2 = \frac{3}{5} \Rightarrow V_2 \sin\theta_2 = 60\cdot\frac{3}{5} = 36.$$

Hence the condition (6) is fulfilled, and so the particles must collide.

From (5) $T = \dfrac{X}{V_1 \cos\theta_1 + V_2 \cos\theta_2} \Rightarrow T = \dfrac{150}{45\cdot\sqrt{1-16/25} + 60\cdot\sqrt{1-9/25}} \Rightarrow T = 2s.$

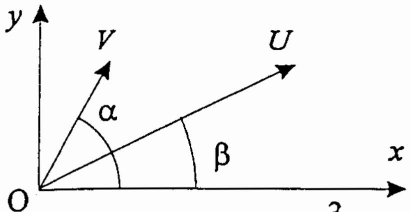
Furthermore $t = T = 2s \Rightarrow$ from (2)

$$y = 45\cdot\frac{4}{5}\cdot 2 - \frac{g\cdot 4}{2} \Rightarrow y = (72-2g)m \Rightarrow y = 52m.$$

7 Solution

Let the second particle be projected with speed U at an angle of elevation β.

Axes and origin:



$V = 40\,ms^{-1}$, $\tan\alpha = \dfrac{3}{4}$

Initial conditions:

the first particle when $t = 0$ the second particle when $t = 2$

$$x_1 = 0,\ y_1 = 0, \qquad x_2 = 0,\ y_2 = 0,$$

$$\dot{x}_1 = V\cos\alpha, \qquad \dot{x}_2 = U\cos\beta,$$

$$\dot{y}_1 = V\sin\alpha, \qquad \dot{y}_2 = U\sin\beta.$$

Equations of motion:

$$\ddot{x}_1 = 0,\ \ddot{y}_1 = -g \qquad \ddot{x}_2 = 0,\ \ddot{y}_2 = -g$$

$$\Rightarrow \dot{x}_1 = V\cos\alpha, \qquad \Rightarrow \dot{x}_2 = U\cos\beta,$$

$$\dot{y}_1 = V\sin\alpha - g\cdot t, \qquad \dot{y}_2 = U\sin\beta - g(t-2).$$

Hence after t seconds the particles are at the positions

$$x_1 = V\cos\alpha\cdot t \quad (1) \qquad \Rightarrow x_2 = U\cos\beta\cdot(t-2) \quad (3)$$

$$y_1 = V\sin\alpha\cdot t - gt^2/2 \quad (2) \qquad y_2 = U\sin\beta(t-2) - \frac{g(t-2)^2}{2}. \quad (4)$$

As the particles collide, their coordinates are equal to each other. Hence

$$t = 3,\ x_1 = x_2 \Rightarrow \text{ from (1) and (3) } 3V\cos\alpha = U\cos\beta. \quad (5)$$

$$t = 3,\ y_1 = y_2 \Rightarrow \text{ from (2) and (4) } 3V\sin\alpha - 4g = U\sin\beta. \quad (6)$$

Dividing (6) and (5), we get $\tan\beta = \tan\alpha - \dfrac{4g}{3}\cdot\dfrac{1}{V\cos\alpha}$. But $\tan\alpha = \dfrac{3}{4} \Rightarrow$

$$\left(1-\cos^2\alpha\right) = \frac{9}{16}\cdot\cos^2\alpha \Rightarrow \cos\alpha = \frac{4}{5}.$$

Hence $\tan\beta = \frac{3}{4} - \frac{4\cdot g}{3}\cdot\frac{1}{40\cdot 4/5} \Rightarrow \tan\beta = \frac{1}{3}$.

From (5) $U = \frac{3V\cos\alpha}{\cos\beta}$. But $\tan\beta = \frac{1}{3} \Rightarrow \cos\beta = \frac{3}{\sqrt{10}}$.

Hence $U = 3\cdot 40\cdot\frac{4}{5}\cdot\frac{\sqrt{10}}{3} \Rightarrow U = 32\cdot\sqrt{10}\,ms^{-1}$.

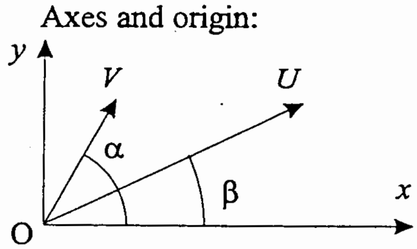
So the second particle was projected with speed $32\cdot\sqrt{10}\,ms^{-1}$ at an angle of elevation $\beta = \tan^{-1}\frac{1}{3}$.

8 Solution

Initial conditions:

the first particle when $t = 0$	the second particle when $t = 1$
$x_1 = 0,\ y_1 = 0$,	$x_2 = 0,\ y_2 = 0$,
$\dot{x}_1 = V\cos\alpha$,	$\dot{x}_2 = U\cos\beta$,
$\dot{y}_1 = V\sin\alpha$,	$\dot{y}_2 = U\sin\beta$,

Axes and origin:



$V = \frac{150}{7}ms^{-1},\ U = \frac{225}{7}ms^{-1}$,

$\tan\alpha = \frac{4}{3},\ \tan\beta = \frac{3}{4}$.

$V = \frac{150}{7}ms^{-1},\ U = \frac{225}{7}ms^{-1}$,

$\tan\alpha = \frac{4}{3},\ \tan\beta = \frac{3}{4}$.

After t seconds the particles are at the positions

$x_1 = V\cos\alpha\cdot t,$ (1) $\qquad x_2 = U\cos\beta\cdot(t-1),$ (3)

$$y_1 = V\sin\alpha \cdot t - gt^2/2, \qquad (2) \qquad y_2 = U\sin\beta(t-1) - \frac{g(t-1)^2}{2}. \qquad (4)$$

Let us first find the values of $\cos\alpha$, $\sin\alpha$ and $\cos\beta$, $\sin\beta$:

$\tan\alpha = \frac{4}{3} \Rightarrow \sin^2\alpha = \frac{16}{9}\left(1-\sin^2\alpha\right) \Rightarrow \sin\alpha = \frac{4}{5}$, hence

$\cos\alpha = \sqrt{1-\sin^2\alpha} \Rightarrow \cos\alpha = \frac{3}{5}$.

Analogously, $\tan\beta = \frac{3}{4} \Rightarrow \sin\beta = \frac{3}{5}, \cos\beta = \frac{4}{5}$.

The particles collide, if and only if at a certain moment their coordinates are equal.

Equate x_1 to x_2.

$$x_1 = x_2 \Rightarrow V\cos\alpha \cdot t = U\cos\beta \cdot (t-1) \Rightarrow t = \frac{U\cos\beta}{U\cos\beta - V\cos\alpha} \Rightarrow$$

$$t = \frac{1}{1 - V\cos\alpha / U\cos\beta} \Rightarrow$$

$t = \dfrac{1}{1 - \dfrac{(150/7)\cdot(3/5)}{(225/7)\cdot(4/5)}} \Rightarrow t = 2s$. If $t = 2 \Rightarrow$ from (2)

$$y_1 = \frac{150}{7}\cdot\frac{4}{5}\cdot 2 - \frac{10\cdot 4}{2} \Rightarrow y_1 = \frac{100}{7}m$$

and from (4) $y_2 = \frac{225}{7}\cdot\frac{3}{5} - \frac{10}{2} \Rightarrow y_2 = \frac{100}{7}m$.

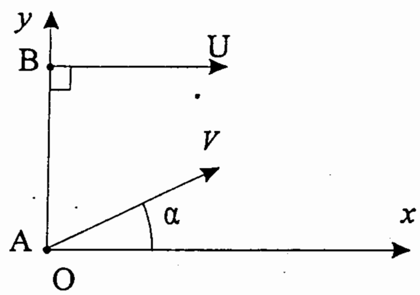
Hence, when $t = 2$, we have $x_1 = x_2$ and $y_1 = y_2$. So the particles must collide at the moment $t = 2$. In other words they collide one second after projection of the second particle.

9 Solution

Initial conditions:

A	B
the first particle when $t=0$	the second particle when $t=0$
$x_1=0,\ y_1=0,$	$x_2=0,\ y_2=h,$
$\dot{x}_1=V\cos\alpha,$	$\dot{x}_2=U,$
$\dot{y}_1=V\sin\alpha,$	$\dot{y}_2=0,$

Axes and origin:



$U=30ms^{-1},\ h\equiv AB=45m$

After t seconds the particles are at positions.

A:
$$x_1=V\cos\alpha\cdot t, \qquad (1)$$
$$y_1=V\sin\alpha\cdot t-\frac{gt^2}{2}, \qquad (2)$$

B:
$$x_2=Ut, \qquad (3)$$
$$y_2=h-\frac{gt^2}{2}. \qquad (4)$$

(a) As B reaches the ground, $y_2=0$;

$$y_2=0\Rightarrow \text{from (4) } t=\left(\frac{2h}{g}\right)^{1/2}\Rightarrow t=\left(\frac{2\cdot 45}{10}\right)^{1/2}\Rightarrow t=3s.$$

Let R be the horizontal distance travelled by B. Hence $t=3,\ x_2=R\Rightarrow$ from (3) $R=U\cdot 3\Rightarrow R=30\cdot 3=90m$.

(b) $t=3,\ x_1=R\Rightarrow$ from (1) $\quad 3V\cos\alpha=R;\qquad (5)$

$t=3,\ y_1=0\Rightarrow$ from (2) $\quad 3V\sin\alpha=\frac{9}{2}g.\qquad (6)$

Dividing (6) by (5), $\tan\alpha=\frac{9g}{2R}\Rightarrow\tan\alpha=\frac{9\cdot 10}{2\cdot 90}\Rightarrow\tan\alpha=\frac{1}{2}\Rightarrow\alpha=\tan^{-1}\frac{1}{2};$

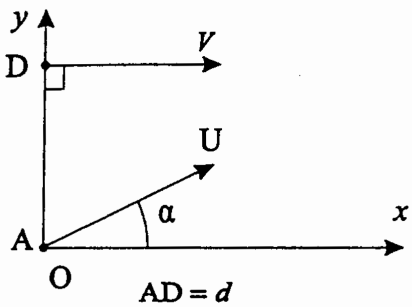
$t = 3,\ x_1 = \mathrm{R} \Rightarrow$ from (1) $V = \dfrac{\mathrm{R}}{3\cos\alpha}$. But $\tan\alpha = \dfrac{1}{2}$, hence

$\left(1 - \cos^2\alpha\right) = \dfrac{1}{4}\cos^2\alpha \Rightarrow$

$\cos\alpha = \dfrac{2}{\sqrt{5}}$. Hence $V = \dfrac{90}{3}\cdot\dfrac{\sqrt{5}}{2} \Rightarrow V = 15\sqrt{5}\,ms^{-1}$.

10 Solution

Axes and origin:



Initial conditions.

The particle from A	The particle from B
when $t = 0$	
$x_1 = 0,\ y_1 = 0$,	$x_2 = 0,\ y_2 = d$,
$\dot{x}_1 = U\cos\alpha$,	$\dot{x}_2 = V$,
$\dot{y}_1 = U\sin\alpha$,	$\dot{y}_2 = 0$.

After t seconds the particles are at positions

$x_1 = U\cos\alpha\cdot t,$ (1) $\qquad x_2 = V\,t,$ (3)

$y_1 = U\sin\alpha\cdot t - \dfrac{gt^2}{2},$ (2) $\qquad y_2 = d - \dfrac{g\,t^2}{2},$ (4)

(a) The particles do collide at a time T if and only if at this moment $x_1 = x_2$ and $y_1 = y_2$. Hence

$$t = T,\ x_1 = x_2 \Rightarrow \text{from (1) and (3) } V = U\cos\alpha. \qquad (5)$$

$$t = T,\ y_1 = y_2 \Rightarrow \text{from (2) and (4) } U\sin\alpha\cdot T = d. \qquad (6)$$

And (6) is the second condition which must also be satisfied.

(b) $\tan\alpha = \frac{8}{15} \Rightarrow \sin^2\alpha = \frac{64}{225}\left(1-\sin^2\alpha\right) \Rightarrow \sin\alpha = \frac{8}{17}$.

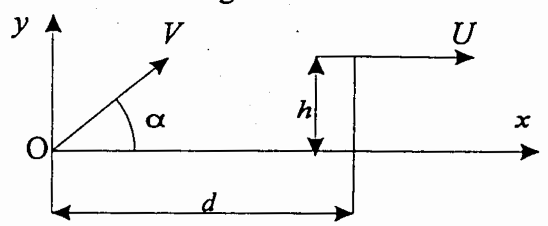
Hence $\cos\alpha = \left(1-\frac{64}{289}\right)^{1/2} \Rightarrow \cos\alpha = \frac{5}{17}$.

Now we can check (5): $45 = 51\cdot\frac{15}{17} \Rightarrow 45 = 45$. Hence (5) is fulfilled, and so the particles do collide. From (6) we find the time of collision T: $T = \frac{d}{U\sin\alpha} \Rightarrow T = \frac{60}{51\cdot 8/17} \Rightarrow T = 2{,}5\,s$.

Let h be the height at which the particles collide. Hence $t = T, y_2 = h \Rightarrow$ from (4) $h = d - \frac{gT^2}{2} \Rightarrow h = 60 - \frac{10\cdot 6{,}25}{2} \Rightarrow h = 28{,}75\,m$.

11 Solution

Axes and origin:



Initial conditions when $t = 0$:

projectile	target
$x_1 = 0,\ y_1 = 0,$	$x_2 = d,\ y_2 = h,$
$\dot{x}_1 = V\cos\alpha,$	$\dot{x}_2 = U,$
$\dot{y}_1 = V\sin\alpha,$	$\dot{y}_2 = 0.$

Equations of motion

$$x_1 = V\cos\alpha \cdot t, \qquad (1) \qquad\qquad x_2 = d + Ut, \qquad (3)$$

$$y_1 = V\sin\alpha \cdot t - \frac{gt^2}{2}, \qquad (2) \qquad\qquad y_2 = h, \qquad (4)$$

(a) Let T_m be the time when the projectile reaches its greatest height. Then $t = T_m$,

$\dot{y}_1 = 0 \Rightarrow$ from (2) $V\sin\alpha - gT_m = 0 \Rightarrow T_m = \dfrac{V\sin\alpha}{g}$;

$t = T_m$, $y_1 = 2h \Rightarrow$ from (2)

$$2h = V\sin\alpha \cdot T_m - \frac{gT_m^2}{2} \Rightarrow 2h = \frac{(V\sin\alpha)^2}{g} - \frac{g}{2}\left(\frac{V\sin\alpha}{g}\right)^2 \Rightarrow$$

$h = \dfrac{V^2\sin^2\alpha}{g}$. Let T be the time when the projectile reaches the height h. Hence

$t = T$, $y_1 = h \Rightarrow$ from (2) $V\sin\alpha \cdot T - \dfrac{gT^2}{2} = h \Rightarrow T^2 - \dfrac{2V\sin\alpha}{g}T + \dfrac{2h}{g} = 0$;

$T = \dfrac{V\sin\alpha}{g} \pm \sqrt{\dfrac{V^2\sin^2\alpha}{g^2} - \dfrac{2h}{g}}$, but $h = \dfrac{V^2\sin^2\alpha}{g}$, so we have

$$T = \frac{V\sin\alpha}{g}\left(1 - \frac{1}{\sqrt{2}}\right) \text{ or}$$

$$T = \frac{V\sin\alpha}{g}\left(1 + \frac{1}{\sqrt{2}}\right).$$

Here $T_- := \dfrac{V\sin\alpha}{g}\left(1 - \dfrac{1}{\sqrt{2}}\right)$ is the time when the projectile just clears the wall and

$T_+ := \dfrac{V\sin\alpha}{g}\left(1 + \dfrac{1}{\sqrt{2}}\right)$ is the time of collision. Hence $t = T_-$, $x_1 = d \Rightarrow$ from (1)

$$d = V\cos\alpha \cdot T_- \Rightarrow d = V\cos\alpha \cdot \frac{V\sin\alpha}{d}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right);\ d = 6\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \Rightarrow d = 3\left(2-\sqrt{2}\right)m.$$

(b) $t = T_+, x_1 = x_2 \Rightarrow$ from (1) and (3) $V\cos\alpha \cdot T_+ = d + u \cdot T_+$, $u = V\cos\alpha - \dfrac{d}{T_+}$ and

from (a)

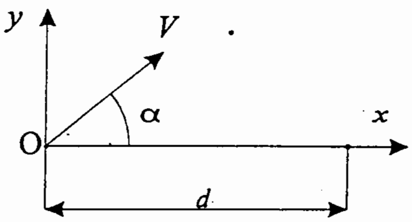
$$u = V\cos\alpha - \frac{V^2\cos\alpha\sin\alpha}{g}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\frac{1}{((V\sin\alpha)/g)\cdot\left((\sqrt{2}+1)/\sqrt{2}\right)};$$

$$u = V\cos\alpha - V\cos\alpha\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right); u = V\cos\alpha\cdot\frac{2}{\sqrt{2}+1}; u = 2V\cos\alpha\left(\sqrt{2}-1\right);$$

$$u = 10\left(\sqrt{2}-1\right)ms^{-1}.$$

12 Solution

Axes and origin: (a) Equations of motion



$$x = V\cos\alpha\cdot t, \quad (1)$$

$$y = V\sin\alpha\cdot t - \frac{gt^2}{2}. \quad (2)$$

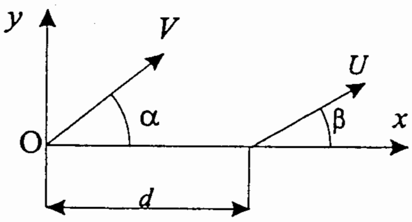
Let the time of collision be T. Hence :

$t = T,\ x = d \Rightarrow$ from (1) $d = V\cos\alpha\cdot T \Rightarrow T = \dfrac{d}{V\cos\alpha}$;

$t = T,\ y = 0 \Rightarrow$ from (2) $0 = V\sin\alpha\cdot T - \dfrac{gT^2}{2} \Rightarrow V = \dfrac{gT}{2\sin\alpha}$

$$\Rightarrow V = \frac{g}{2\sin\alpha}\cdot\frac{d}{V\cos\alpha} \Rightarrow V = \left(\frac{d\,g}{\sin 2\alpha}\right)^{1/2}.$$

Axes and origin: (b) Equations of motion



projectile target

$$x_1 = V\cos\alpha\cdot t, \quad (1)$$

$$y_1 = V\sin\alpha\cdot t - \frac{gt^2}{2}, \quad (2)$$

$$x_2 = U\cos\beta\cdot t + d, \quad (3)$$

$$y_2 = U\sin\beta\cdot t - \frac{gt^2}{2}. \quad (4)$$

At the time of collision the coordinates of the projectile and the target are equal.

Hence

$x_1 = x_2 \Rightarrow$ from (1) and (3) $V\cos\alpha \cdot t = U\cos\beta \cdot t + d \Rightarrow t = \dfrac{d}{V\cos\alpha - U\cos\beta}$; (5)

$y_1 = y_2 \Rightarrow$ from (2) and (4) $V\sin\alpha = U\sin\beta \Rightarrow V = \dfrac{U\sin\beta}{\sin\alpha}$. Substituting this into (5), we get

$$t = \frac{d\sin\alpha}{U(\cos\alpha\sin\beta - \cos\beta\sin\alpha)};\ t = \frac{d\sin\alpha}{U}\cdot\frac{1}{\sin(\beta-\alpha)}.$$

Exercise 7.4

1 Solution

$a = \frac{v^2}{r}$ is the observed acceleration.

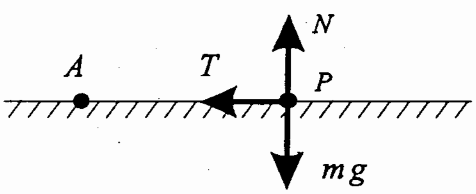
(a) $v = 8,\ r = 2 \Rightarrow a = \frac{8^2}{2} \Rightarrow a = 32\ ms^{-2}$.

(b) The resultant force F is $ma = \frac{mv^2}{r}$. Hence $F = \frac{mv^2}{r} \Rightarrow m = \frac{rF}{v^2}$. So we have

$v = 3, r = 6, F = 6 \Rightarrow m = \frac{6 \cdot 6}{3^2} \Rightarrow m = 4\,kg$.

2 Solution

Forces on P



Here T is the tension in the inextensible string whose length is $l = 0{,}5\,\text{m}$, and N is the reaction force. Observed acceleration is $a = \frac{v^2}{l}$. Hence the vector sum of forces on P is $\frac{mv^2}{l}$ towards A.

(a) The resultant has vertical component zero $\Rightarrow N = mg$.

$m = 0{,}25\,kg \Rightarrow N = \frac{1}{4}g$. The resultant has horizontal component

$ma = \frac{mv^2}{l} \Rightarrow T = \frac{mv^2}{l}$;

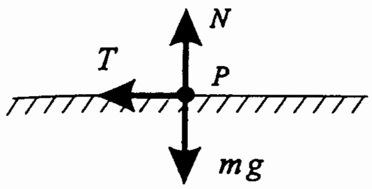
$m = 0{,}25,\ l = 0{,}5,\ v = 8 \Rightarrow T = \frac{0{,}25 \cdot 64}{0.5} \Rightarrow T = 32\,\text{N}$.

(b) The string breaks if $T > 50\,\text{N}$. Hence from (a) $\frac{mv^2}{l} \le 50 \Rightarrow v \le \left(\frac{50 \cdot l}{m}\right)^{1/2}$. But

$\omega = \frac{v}{l} \Rightarrow\ \omega \le \left(\frac{50}{m \cdot l}\right)^{1/2}$; $m = 0{,}25,\ l = 0{,}5 \Rightarrow \omega \le 20\,rad\ s^{-1}$.

3 Solution

Forces on P



Observed acceleration is $a = ml\omega^2$, where $\omega = 2\pi\, rad\ s^{-1}$, and $l = 2\,\text{m}$ is the length of string.

(a) The resultant has horizontal component

$ma = ml\omega^2 \Rightarrow T = ml\omega^2$;

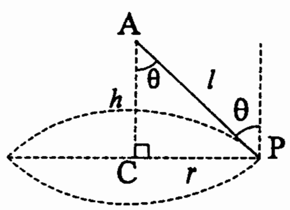
$m = 2,\ l = 2,\ \omega = 2\pi \Rightarrow T = 16\pi^2\,\text{N}$.

(b) The string breaks if $T > 20g$. But from (a) $T = ml\omega^2 \Rightarrow T = \dfrac{mv^2}{l}$. Hence

$$\frac{mv^2}{l} \le 20g \Rightarrow\ v \le \left(\frac{20 \cdot g \cdot l}{m}\right)^{1/2};\ l = 2,\ m = 2 \Rightarrow v \le (20g)^{1/2}.$$

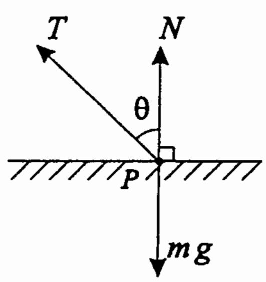
4 Solution

Dimension diagram



$l = 1\,\text{m}, h = 0{,}5\,\text{m},\ \omega = \dfrac{2\pi}{2} = \pi\ rad\ s^{-1}$.

Forces on P



Observed acceleration is $a = r\omega^2$, where $r = \sqrt{l^2 - h^2}$. Hence the vector sum of forces on P is $ma = mr\omega^2$. The resultant has a horizontal component

$mr\omega^2 \Rightarrow T\sin\theta = mr\omega^2 \Rightarrow\ T = \dfrac{mr\omega^2}{\sin\theta}$. But $\sin\theta = \dfrac{r}{l} \Rightarrow T = m\omega^2 l$;

$m = 1, l = 1, \omega = \pi \Rightarrow T = \pi^2\,\text{N}$.

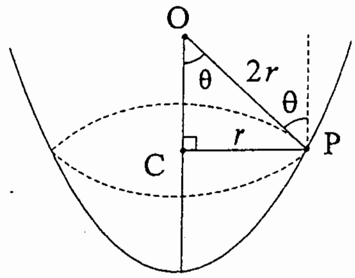
The resultant has a vertical component zero $\Rightarrow T\cos\theta + N = mg \Rightarrow$ from (a)

$N = mg - m\omega^2 l\cos\theta$. But $\cos\theta = \dfrac{h}{l} \Rightarrow N = mg - m\omega^2 h$;

$m = 1, h = 0{,}5, \omega = \pi \Rightarrow N = g - \dfrac{\pi^2}{2}\text{N}$.

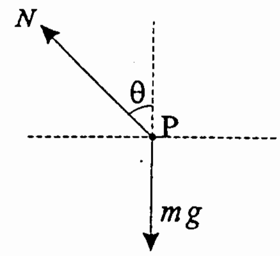
5 Solution

Dimension diagram



O is the center of the hemispherical bowl

Forces on P



N is normal to the surface at P, hence directed towards O.

P performs uniform circular motion about C, hence the resultant force is directed towards C.

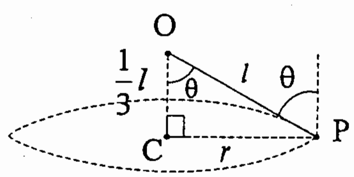
The resultant has a vertical component zero $\Rightarrow N\cos\theta = mg$. (1)

The resultant has a horizontal component $mr\omega^2 \Rightarrow N\sin\theta = mr\omega^2$. (2)

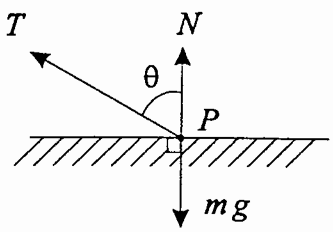
Dividing (2) by (1), $\dfrac{r\omega^2}{g} = \tan\theta$. But $\tan\theta = \dfrac{r}{\sqrt{4r^2 - r^2}} = \dfrac{1}{\sqrt{3}} \Rightarrow \omega^2 = \dfrac{g}{r\sqrt{3}}$.

6 Solution

Dimension diagram



Forces on P



Observed acceleration is $a = \dfrac{v^2}{r}$, where

$r = \sqrt{l^2 - l^2/9} = \left(\sqrt{8}/3\right)\cdot l$. Hence the vector sum of forces on P is $ma = \dfrac{mv^2}{r}$ and directed towards C.

(a) The resultant has a horizontal component $\dfrac{mv^2}{r} \Rightarrow T\sin\theta = \dfrac{mv^2}{r} \Rightarrow T = \dfrac{mv^2}{r\sin\theta}$.

But $\sin\theta = \dfrac{r}{l} \Rightarrow T = \dfrac{9mv^2}{8l}$.

The resultant has a vertical component zero

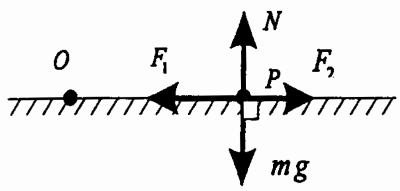
$$\Rightarrow T\cos\theta + N = mg \Rightarrow N = mg - \frac{9mv^2}{8l}\cos\theta .$$

But $\cos\theta = \frac{1}{3} \Rightarrow N = m\left(g - \frac{3v^2}{8l}\right)$.

(b) As $N \geq 0$, then from (a) $m\left(g - \frac{3v^2}{8l}\right) \geq 0 \Rightarrow v^2 \leq \frac{8gl}{3}$.

7 Solution

Forces on P



$F_1 = 8v,\ F_2 = \frac{k}{r^2}$,

$OP = r$.

Observed acceleration is $a = \frac{v^2}{r}$.

Hence the vector sum of forces is

$ma = \frac{mv^2}{r}$ and directed to O.

(a) The horizontal component of the resultant force is $\frac{mv^2}{r} \Rightarrow F_1 - F_2 = \frac{mv^2}{r} \Rightarrow 8v - \frac{k}{r^2} = \frac{mv^2}{r} \Rightarrow$

$$v^2 - \frac{8r}{m}v + \frac{k}{mr} = 0 \Rightarrow v_{\pm} = \frac{4r}{m} \pm \sqrt{\left(\frac{4r}{m}\right)^2 - \frac{km}{r}};$$

$k = 75, r = 1, m = 0{,}2 \Rightarrow v_+ = 25$ and $v_- = 15$.

(b) $T = \frac{2\pi r}{v} \Rightarrow r = \frac{Tv}{2\pi}$;

$$T = \frac{\pi}{5}, v = 20 \Rightarrow r = \frac{\pi}{5}\cdot\frac{20}{2\pi} \Rightarrow r = 2.$$

From (a) $8v - \frac{k}{r^2} = \frac{mv^2}{r} \Rightarrow\ k = 8vr^2 - mv^2r$; $r = 2, v = 20, m = 0{,}2 \Rightarrow k = 480$.

(c) From (b) $k = 8vr^2 - mv^2r$; $r = 1, m = 0{,}2 \Rightarrow k = 8v - 0{,}2v^2$. The function $k(v) = 8v - 0{,}2v^2$ has the derivative $k'(v) = 8 - 0{,}4v$. Hence $k'(v) = 0 \Rightarrow v = 20$.

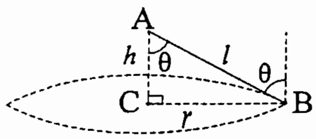
And $k''(v) = -0{,}4 < 0 \Rightarrow$ at the point $v = 20$ the function $k(v)$ has its maximum value $k(20) = 80$.

So $0 \leq k \leq 80$.

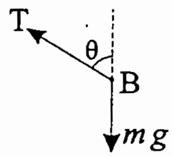
Exercise 7.5

1 Solution

Dimension diagram



Forces on B



$l = 2\,\text{m},\ h = 1\,\text{m}$

The resultant force on B is horizontal towards C of magnitude $mr\omega^2$. The resultant has a vertical component zero

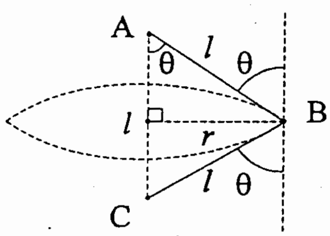
$\Rightarrow T\cos\theta = mg$. But $\cos\theta = \dfrac{h}{l} \Rightarrow T = \dfrac{mgl}{h}$; $l = 2,\ h = 1,\ m = 6 \Rightarrow T = 12g$.

The resultant has a horizontal component $mr\omega^2 \Rightarrow T\sin\theta = mr\omega^2$. But

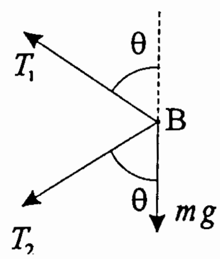
$\sin\theta = \dfrac{r}{l}$ and $T = \dfrac{mgl}{h}$ $\Rightarrow \omega^2 = \dfrac{g}{h}$; $h = 1 \Rightarrow \omega = \sqrt{g}$.

2 Solution

Dimension diagram



Forces on B



The resultant force on B is $mr\omega^2$ horizontally to the left.

(a) The sum of the horizontal components is

$mr\omega^2 \Rightarrow T_1\sin\theta + T_2\sin\theta = mr\omega^2$. But

$$\sin\theta = \frac{r}{l} \Rightarrow \ T_1 + T_2 = ml\omega^2. \qquad (1)$$

The sum of vertical components is zero$\Rightarrow T_1\cos\theta = T_2\cos\theta + mg$. But

$$\cos\theta = \frac{1}{2} \Rightarrow T_1 - T_2 = 2mg. \qquad (2)$$

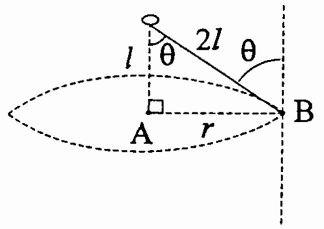
$$(1) + (2) \Rightarrow T_1 = m\left(\frac{l\omega^2}{2} + g\right).$$

From (1) $\Rightarrow T_2 = ml\omega^2 - T_1 \Rightarrow T_2 = m\left(\frac{l\omega^2}{2} - g\right).$

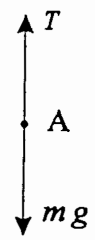
(b) We have $T_1 > T_2$. Hence $T_2 > 0 \Rightarrow \frac{l\omega^2}{2} - g > 0 \Rightarrow \omega > \left(\frac{2g}{l}\right)^{1/2}$.

3 Solution

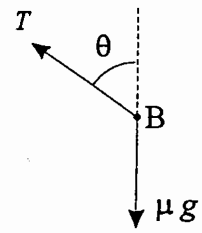
Dimension diagram



Forces on B



Forces on A



(a) The resultant force on A is zero $\Rightarrow T = mg$ (1)

The resultant force on B is $\mu r\omega^2$ horizontally to the left. The sum of vertical components zero $\Rightarrow T\cos\theta = \mu g$. But $\cos\theta = \frac{1}{2} \Rightarrow T = 2\mu g$. Using (1), $m = 2\mu$.

(b) The sum of horizontal components is $\mu r\omega^2$. Hence $T\sin\theta = \mu r\omega^2$.

But $\sin\theta = \frac{r}{2l}$ and

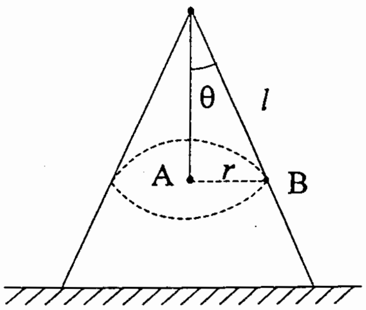
$T = 2\mu g \Rightarrow \omega^2 = \frac{g}{l} \Rightarrow \omega = \sqrt{\frac{g}{l}}$.

4 Solution

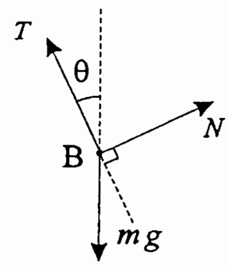
Dimension diagram

Forces on B

Forces on A



$\theta = \frac{\pi}{6}$



(a) The resultant force on A is zero $\Rightarrow T = mg$.

The resultant force on B is $mr\omega^2$ horizontally to the left. Its vertical component is zero $\Rightarrow T\cos\theta + N\sin\theta = mg \;\Rightarrow N = mg\left(2-\sqrt{3}\right)$.

(b) Its horizontal component is

$mr\omega^2 \Rightarrow T\sin\theta - N\cos\theta = mr\omega^2 \Rightarrow r\omega^2 = g\left(2-\sqrt{3}\right)$.

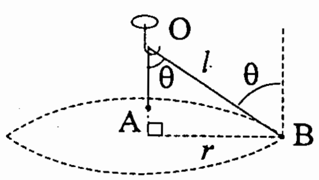
But $r = \frac{l}{2} \Rightarrow \omega = \left(\frac{2g}{l}\left(2-\sqrt{3}\right)\right)^{1/2}$.

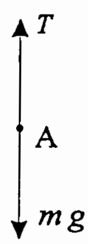
5 Solution

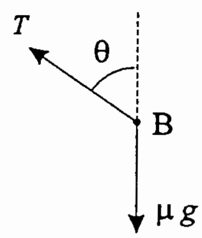
Dimension diagram

Forces on A

Forces on B







(a) The resultant force on A is zero $\Rightarrow T = mg$.

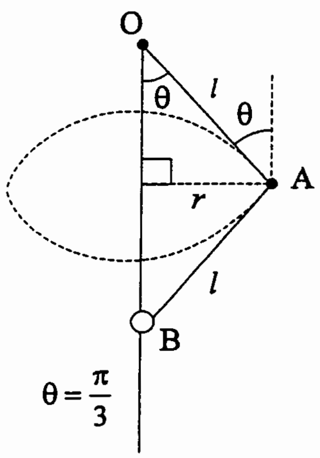
The resultant force on B is $\frac{\mu v^2}{r}$ horizontally to the left. Its vertical component is zero $\Rightarrow T\cos\theta = \mu g \Rightarrow \cos\theta = \frac{\mu g}{T} \Rightarrow \theta = \cos^{-1}\left(\frac{\mu}{m}\right)$. Its horisontal component is $\frac{\mu v^2}{r} \Rightarrow T\sin\theta = \frac{\mu v^2}{r}$. But

$$r = l\sin\theta \Rightarrow l = \frac{\mu v^2}{T\sin^2\theta} \Rightarrow \ l = \frac{\mu v^2}{mg\left(1-\frac{\mu^2}{m^2}\right)} \Rightarrow \ l = \frac{m\mu v^2}{g\left(m^2-\mu^2\right)}.$$

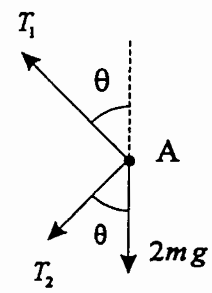
(b) $r = l\sin\theta \Rightarrow r = \frac{\mu v^2}{T\sin\theta} \Rightarrow r = \frac{\mu v^2}{mg\sqrt{1-\frac{\mu^2}{m^2}}} \Rightarrow r = \frac{\mu v^2}{g\sqrt{m^2-\mu^2}}$.

6 Solution

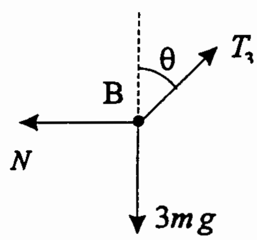
Dimension diagram



Forces on B



Forces on A



AB is a light inextensible string, therefore $T_2 = T_3$.

B is not moving, hence the resultant force on it is zero. Therefore for the vertical component of the resultant we have

$T_3\cos\theta = 3mg \Rightarrow T_3 = \frac{3mg}{\cos\theta} \Rightarrow T_3 = 6mg$.

The resultant force on A is $2mr\omega^2$ directad to the left. Its vertical

component is zero $\Rightarrow T_1\cos\theta = T_2\cos\theta + 2mg \Rightarrow T_1 = T_2 + \frac{2mg}{\cos\theta}$. But

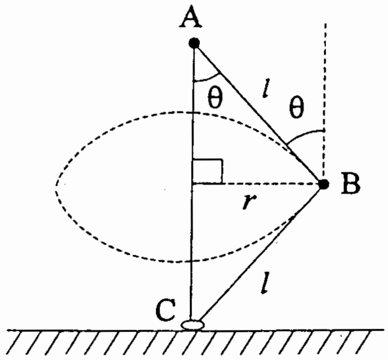
$T_2 = T_3 = \frac{3mg}{\cos\theta} \Rightarrow T_1 = \frac{5mg}{\cos\theta} \Rightarrow T_1 = 10mg.$

The resultant on A has the horisontal component $2mr\omega^2$, hence

$T_1\sin\theta + T_2\sin\theta = 2mr\omega^2$. But $r = l\sin\theta \Rightarrow \omega^2 = \frac{T_1+T_2}{2m}\cdot\frac{1}{l} \Rightarrow \omega = \sqrt{\frac{8g}{l}}$.

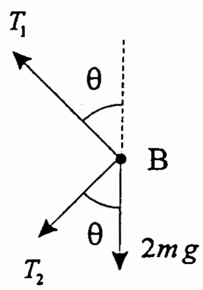
7 Solution

Dimension diagram

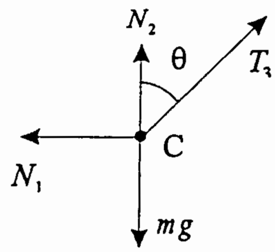


$l = 2\,\mathrm{m}$, $AC = 2\sqrt{3}\,\mathrm{m}$

Forces on B



Forces on C



$m = 1\,kg$

The resultant force on B is $2mr\omega^2$ to the left. Its horizontal component is $2mr\omega^2$, hence $T_1 \sin\theta + T_2 \sin\theta = 2mr\omega^2$, where $r = l\sin\theta$. Its vertical component is zero, hence $T_1 \cos\theta = T_2 \cos\theta + 2mg$, where $\cos\theta = \dfrac{\sqrt{3}}{2}$. So we have two equations:

$$T_1 + T_2 = 2ml\omega^2, \tag{1}$$

$$T_1 - T_2 = \frac{4mg}{\sqrt{3}}\ ; \tag{2}$$

(1) + (2) $\Rightarrow T_1 = ml\omega^2 + \dfrac{2mg}{\sqrt{3}} \Rightarrow T_1 = 2\omega^2 + \dfrac{2g}{\sqrt{3}}$;

(1) - (2) $\Rightarrow T_2 = ml\omega^2 - \dfrac{2mg}{\sqrt{3}} \Rightarrow T_2 = 2\omega^2 - \dfrac{2g}{\sqrt{3}}$.

The rod BC is light, hence $T_3 = T_2$. The resultant force on C is zero. So for its vertical component we have

$$N_2 + T_3\cos\theta = mg \Rightarrow N_2 = g - \left(2\omega^2 - \frac{2g}{\sqrt{3}}\right)\cdot\frac{\sqrt{3}}{2} \Rightarrow$$

$$N_2 = 2g - \sqrt{3}\,\omega^2. \tag{3}$$

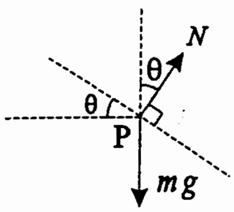
(b) If the ring rests on a ledge, then $N_2 \geq 0$ and hence from (3) $\omega^2 \leq \dfrac{2g}{\sqrt{3}}$.

So if $\omega^2 > \dfrac{2g}{\sqrt{3}}$ the ring lifts off the ledge.

Exercise 7.6

1 Solution

Forces on the car P. Let R be the radius of the track: $R = 10\,\text{m},\ \theta = 12°$.



The vertical components sum to zero $\Rightarrow N\cos\theta = mg$.

(1)

The horizontal components sum to $\dfrac{mv^2}{R} \Rightarrow N\sin\theta = \dfrac{mv^2}{R}$.

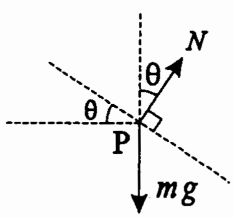
(2)

Dividing (2) by (1), we obtain

$$\tan\theta = \frac{v^2}{Rg} \Rightarrow v = (Rg\tan\theta)^{1/2} \Rightarrow v = 14{,}4\ \text{m}\,s^{-1}.$$

2 Solution

Forces on the car P. Let R be the radius of the track.



The vertical components sum to zero $\Rightarrow N\cos\theta = mg$.

(1)

The horizontal components sum to $\dfrac{mv^2}{R} \Rightarrow N\sin\theta = \dfrac{mv^2}{R}$.

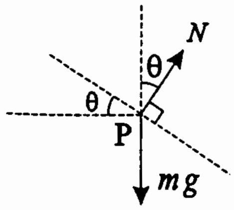
(2)

Dividing (2) by (1), we obtain $\tan\theta = \dfrac{v^2}{Rg}$;

$R = 200, v = 30, g = 9{,}8 \Rightarrow \tan\theta = 0{,}4592 \Rightarrow \theta = 24{,}7°$.

3 Solution

Forces on the car P. Let R be the radius of a bend.



The vertical components sum to zero $\Rightarrow N\cos\theta = mg$.(1)

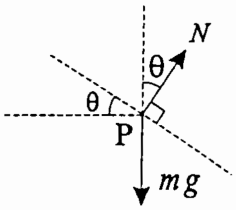
The horizontal components sum to $\dfrac{mv^2}{R} \Rightarrow N\sin\theta = \dfrac{mv^2}{R}$.(2)

Dividing (2) by (1), we obtain $\tan\theta = \dfrac{v^2}{Rg} \Rightarrow\ v = (Rg\cdot\tan\theta)^{1/2}$;

$R = 80,\ g = 9{,}8,\ \theta = 10° \Rightarrow\ v = 11{,}8\ \mathrm{m}\,s^{-1}$.

4 Solution

Forces on the aircraft P. Let R be the radius of a circle.



The vertical components sum to zero $\Rightarrow N\cos\theta = mg$.

(1)

The horizontal components sum to $\Rightarrow\ N\sin\theta = \dfrac{mv^2}{R}$.

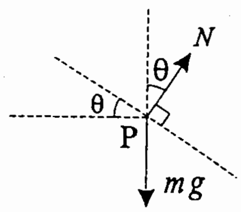
(2)

Dividing (2) by (1), we obtain $\tan\theta = \dfrac{v^2}{Rg}$;

$R = 4000, v = 100, g = 9{,}8 \Rightarrow \tan\theta = 0{,}2551 \Rightarrow \theta = 14{,}3°$.

5 Solution

Forces on cars P. Let θ_1 and θ_2 be the angles of the inner and outer banking respectively.



The vertical components sum to zero $\Rightarrow\ N\cos\theta = mg$.

(1)

The horizontal components sum to $\dfrac{mv^2}{R} \Rightarrow\ N\sin\theta = \dfrac{mv^2}{R}$.

(2)

Dividing (2) by (1), we obtain $\tan\theta = \dfrac{v^2}{Rg}$;

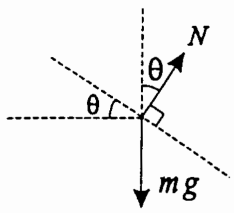
$v = 80\ km\,h^{-1} = 22{,}2\ \mathrm{m}\,s^{-1},\ R = 200\,\mathrm{m},\ g = 9{,}8\ \mathrm{m}\,s^{-2} \Rightarrow \tan\theta_1 = 0{,}25 \Rightarrow \theta_1 = 14°$.

$v = 160\ km\,h^{-1} = 44{,}4\ \mathrm{m}\,s^{-1},\ R = 220\,\mathrm{m},\ g = 9{,}8\ \mathrm{m}\,s^{-2} \Rightarrow \tan\theta_2 = 0{,}916 \Rightarrow \theta_2 = 42{,}4°$.

Hence the difference between the angles of banking is $\theta_2 - \theta_1 = 28{,}4°$.

6 Solution

Forces on the engine. The normal reaction $\vec{N}$ is a reaction to the force the train exerts at right angles to the rail.



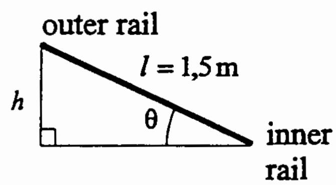
The vertical components sum to zero $\Rightarrow$ $N\cos\theta = mg$. (1)

The horizontal components sum to $\dfrac{m v^2}{R}$, where R is the radius of a circular bend, $\Rightarrow$ $N\sin\theta = \dfrac{m v^2}{R}$ (2)

Dividing (2) by (1), we obtain $\tan\theta = \dfrac{v^2}{Rg}$;

$v = 40\ km\,h^{-1} = 11{,}11\ m\,s^{-1}$, $R = 1000\,m$, $g = 9{,}8\ m\,s^{-2} \Rightarrow \tan\theta = 0{,}0126$.

Dimension diagram

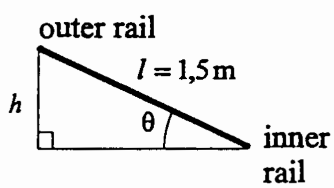


If θ is small $\sin\theta \equiv \theta \equiv \tan\theta$. Hence for a small θ;

$h = l\cdot\tan\theta \Rightarrow h = 1{,}5\cdot 0{,}0126 = 0{,}0189\,m \Rightarrow h = 18{,}9\,mm$.

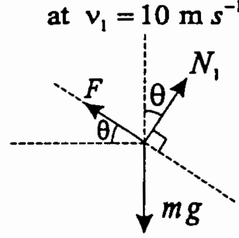
7 Solution

Dimension diagram

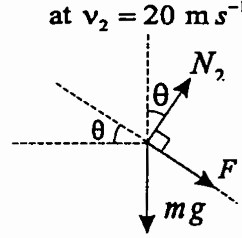


Forces on engine

at $v_1 = 10\ m\,s^{-1}$



at $v_2 = 20\ m\,s^{-1}$



$R = 1000\,m$ is the radius of a circular bend.

In each case, the resultant force is directed horizontally to the centre of the circle with magnitude $\dfrac{m v^2}{R}$, so the vertical components sum to zero, while the horizontal components sum to $\dfrac{m v^2}{R}$.

$N_1\cos\theta + F\sin\theta = mg$, (1) $\qquad N_2\cos\theta - F\sin\theta = mg$, (3)

$$N_1 \sin\theta - F\cos\theta = \frac{m v_1^2}{R}, \qquad (2) \qquad\qquad N_2 \sin\theta + F\cos\theta = \frac{m v_2^2}{R}. \qquad (4)$$

$$(1)\times\sin\theta - (2)\times\cos\theta \Rightarrow \ F = m\left(g\sin\theta - \frac{v_1^2\cos\theta}{R}\right);$$

$$(4)\times\cos\theta - (3)\times\sin\theta \Rightarrow \ F = m\left(\frac{v_2^2\cos\theta}{R} - g\sin\theta\right).$$

Eliminating F, $2g\sin\theta = \left(v_1^2 + v_2^2\right)\dfrac{\cos\theta}{R} \Rightarrow \ \tan\theta = \dfrac{v_1^2 + v_2^2}{2gR} \Rightarrow \ \tan\theta = 0{,}0255$.

But $\sin\theta \equiv \theta \equiv \tan\theta$, for θ small. Hence $h = l\cdot\sin\theta \equiv 1{,}5\cdot\tan\theta \Rightarrow \ h = 38{,}3\,\mathrm{mm}$.

Diagnostic Test 7

1 Solution

Origin and positive direction: choose initial position as origin and initial direction of motion as positive.

Equation of motion: $\ddot{x} = -\dfrac{1}{3v^2}$.

Initial conditions: $t = 0,\ x = 0,\ v = u$.

Relation between x and v: $v\dfrac{dv}{dx} = -\dfrac{1}{3v^2} \Rightarrow dx = -3v^3\,dv \Rightarrow x = -\dfrac{3}{4}v^4 + c$, c constant.

$$x = 0, v = u \Rightarrow c = \frac{3}{4}u^4 \Rightarrow x = \frac{3}{4}\left(u^4 - v^4\right). \qquad (1)$$

Relation between v and t: $\dfrac{dv}{dt} = \dfrac{-1}{3v^2} \Rightarrow dt = -3v^2 dv \Rightarrow t = -v^3 + A$, A constant.

$$t = 0, v = u \Rightarrow A = u^3 \Rightarrow t = \left(u^3 - v^3\right). \qquad (2)$$

When the particle comes to rest its velocity is zero. So $v = 0 \Rightarrow$ from (1) $x = \dfrac{3}{4}u^4$ is the required distance and from (2) $t = u^3$ is the required time.

2 Solution

Choose initial direction of motion as positive.

Equation of motion: $\ddot{x} = -k\,v^3$, $k > 0$ constant.

Initial conditions: $t = 0,\ x = 0,\ v = V$.

(a) Relation between x and v: $v\dfrac{dv}{dx} = -k\,v^3 \Rightarrow dx = -\dfrac{dv}{k\,v^2} \Rightarrow x = \dfrac{1}{k\,v} + c$, c constant.

$$x = 0, v = V \Rightarrow c = -\frac{1}{kV} \Rightarrow x = \frac{1}{k\,v} - \frac{1}{kV} \Rightarrow k\,x = \frac{1}{v} - \frac{1}{V}. \qquad (1)$$

(b) Relation between v and t: $\dfrac{dv}{dt} = -k\,v^3 \Rightarrow dt = -\dfrac{dv}{k\,v^3} \Rightarrow t = \dfrac{1}{2k\,v^2} + A$, A constant.

$t=0, v=V \Rightarrow \quad A=-\frac{1}{2kV^2} \Rightarrow t=\frac{1}{2k}\left(\frac{1}{v^2}-\frac{1}{V^2}\right).$

But from (1) $\frac{1}{v}=kx+\frac{1}{V} \Rightarrow \ t=\frac{1}{2k}\left\{\left(kx+\frac{1}{V}\right)^2-\frac{1}{V^2}\right\} \Rightarrow \ t=\frac{x}{V}+\frac{1}{2}kx^2.$

3 Solution

Choose initial position as origin, and initial direction as positive.

Equation of motion: $m\ddot{x}=-2m-mv \Rightarrow \ddot{x}=-2-v$.

Initial conditions: $t=0,\ x=0,\ v=4$.

Relation between x and v: $v\frac{dv}{dx}=-2-v \Rightarrow \ dx=-\frac{v}{v+2}dv \Rightarrow$

$dx=\left(-1+\frac{2}{v+2}\right)dv \Rightarrow \ x=-v+2\ln|v+2|+c,\ c$ constant.

$x=0, v=4 \Rightarrow c=4-2\ln 6 \Rightarrow \ x=(4-v)-2\ln\frac{6}{v+2}.$

The particle comes to rest $\Rightarrow v=0 \Rightarrow x=4-2\ln 3$.

Relation between v and t: $\frac{dv}{dt}=-2-v \Rightarrow \ dt=-\frac{dv}{v+2} \Rightarrow t=-\ln|v+2|+A,\ A$

constant.

$t=0, v=4 \Rightarrow \ A=\ln 6 \Rightarrow t=\ln\frac{6}{v+2}$. So $v=0 \Rightarrow t=\ln 3$.

4 Solution

Choose initial position as origin, and initial direction of motion as positive.

Equation of motion: $\ddot{x}=-kv^{3/2}$.

Initial conditions: $t=0,\ x=0,\ v=u$.

Relation between v and t: $\frac{dv}{dt}=-kv^{3/2} \Rightarrow \ dt=-\frac{dv}{kv^{3/2}} \Rightarrow t=\frac{2}{k}\cdot\frac{1}{v^{1/2}}+c,\ c$

constant.

$t=0, v=u \Rightarrow \ c=-\frac{2}{k}\cdot\frac{1}{u^{1/2}} \Rightarrow t=\frac{2}{k}\left(\frac{1}{v^{1/2}}-\frac{1}{u^{1/2}}\right)$. From here $t \to +\infty$, as $v \to 0^+$, and

so the particle is never brought to rest.

Relation between x and v: $v\dfrac{dv}{dx}=-k\,v^{3/2}\Rightarrow\; dx=\dfrac{-dv}{k\,v^{1/2}}\Rightarrow\; x=-\dfrac{2\,v^{1/2}}{k}+c,\; c$

constant.

$x=0, v=u\Rightarrow c=\dfrac{2}{k}u^{1/2}\Rightarrow\; x=\dfrac{2}{k}\left(u^{1/2}-v^{1/2}\right)$. From here $x\to\left(\dfrac{2}{k}u^{1/2}\right)^{-}$ as $v\to 0^{+}$.

5 Solution

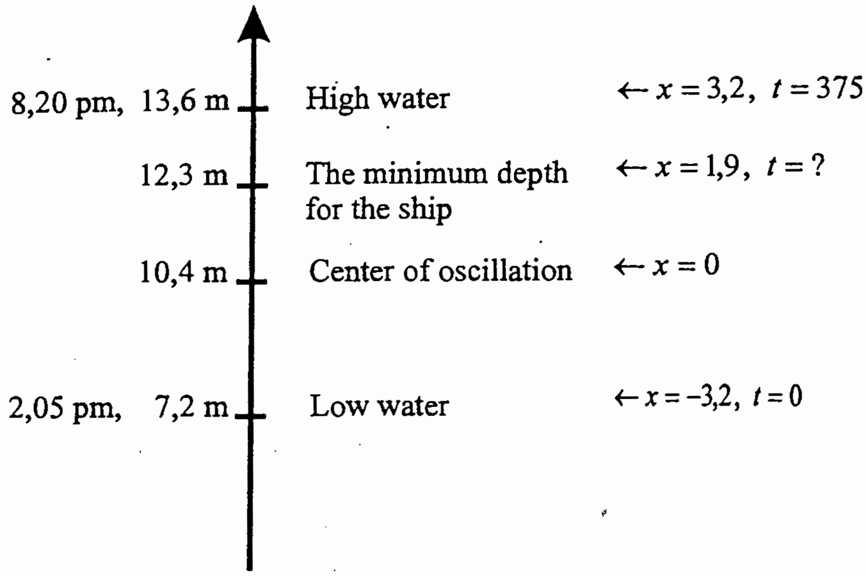
Let l be the length of the pendulum. Hence $T=2\pi\sqrt{\dfrac{l}{9{,}812}}$ and $\tilde{T}=2\pi\sqrt{\dfrac{l}{9{,}921}}$ are the periods of the small oscillations of the pendulum at an old and new places respectively. But $\dfrac{T}{2}=1$, as the pendulum beats exact seconds (each half-oscillation takes one second). Hence $2=2\pi\sqrt{\dfrac{l}{9{,}812}}\Rightarrow l=\dfrac{9{,}812}{\pi^2}\Rightarrow l=0{,}994\,\mathrm{m}$.

(a) $\tilde{T}<T$ hence at the new place the pendulum gains every second by $\dfrac{T-\tilde{T}}{T}$ seconds. So per day it gains $24\cdot 3600\left(1-\dfrac{\tilde{T}}{T}\right)=24\cdot 3600\cdot\left(1-\sqrt{\dfrac{9{,}812}{9{,}921}}\right)=476\,s$.

(b) Let δ be the new length $\Rightarrow T=2\pi\sqrt{\dfrac{\delta}{9{,}921}}\Rightarrow\delta=9{,}921\cdot\dfrac{T^2}{(2\pi)^2}$. But $T=2\Rightarrow$

$\delta=\dfrac{9{,}921}{\pi^2}\Rightarrow\;\delta=1{,}005\,\mathrm{m}$.

6 Solution



Period T is $2 \cdot 375 = 750$ minutes.

Amplitude is $\frac{1}{2}(13{,}6 - 7{,}2) = 3{,}2\,\mathrm{m}$.

Motion is simple harmonic $\Rightarrow \ddot{x} = -n^2 x$, $n = \frac{2\pi}{T} = \frac{\pi}{375}$.

This equation has solution $x = 3{,}2\cos(nt + \alpha)$, $0 \le \alpha < 2\pi$.

Initial conditions: $t = 0$, $x = -3{,}2 \Rightarrow \cos\alpha = -1 \Rightarrow \alpha = \pi \Rightarrow x = -3{,}2\cos nt$.

(a) A minimum depth is $12{,}3\,\mathrm{m}$, if $x = 1{,}9 \Rightarrow 1{,}9 = -3{,}2\cos nt \Rightarrow \; nt = \cos^{-1}\left(-\frac{1{,}9}{3{,}2}\right) \Rightarrow$

$t = \frac{375}{\pi}\left(\pi - \cos^{-1}\frac{19}{32}\right) \; \Rightarrow t = 264\,\text{minutes} = 4{,}24$ and

$T - t = 750 - 264 = 486\,\text{minutes} = 8{,}06$.

Hence the ship can leave the harbour between $2{,}05\,\text{pm} + 4{,}24 = 6{,}29\,\text{pm}$ and

$2{,}05\,\text{pm} + 8{,}06 = 10{,}11\,\text{pm}$.

(b) From (a) on Monday the ship can leave the harbour between

$6{,}29\,\text{pm}$ and $10{,}11\,\text{pm}$. As the period of oscillations is $T = 750\,\text{minutes} = 12{,}30$ the

ship can leave on Tuesday between $6{,}29\,\text{pm} + 12{,}30 = 6{,}59\,\text{am}$ and

$10{,}11\,\text{pm} + 12{,}30 = 10{,}41\,\text{am}$. Acting in such a way we can write the following list:

Monday	6,29 pm — 10,11 pm ,
Tuesday	6,59 am — 10,41 am ,

Tuesday	7,29 pm — 11,11 pm ,
Wednesday	7,59 am — 11,41 am .

Hence the answer is 7,59 am .

7 Solution

Origin is point of release. $\downarrow$ is positive direction $\Rightarrow F = mg - \frac{v^2}{10}$.

Equation of motion: $\ddot{x} = g - \frac{v^2}{5}$.

Initial conditions: $t = 0,\ x = 0,\ v = 0$.

Expression relating v and x: $v\frac{dv}{dx} = g - \frac{v^2}{5} \Rightarrow\ dx = \frac{5v}{5g - v^2}dv \Rightarrow$

$dx = \frac{5}{2}\left(\frac{1}{\sqrt{5g} - v} - \frac{1}{\sqrt{5g} + v}\right)dv \Rightarrow\ x = \frac{5}{2}\left\{\ln\frac{1}{\sqrt{5g} + v} + \ln\frac{1}{\sqrt{5g} - v}\right\} + c,\ c$ constant.

$x = -\frac{5}{2}\ln\left(5g - v^2\right) + c$.

$x = 0,\ v = 0 \Rightarrow c = \frac{5}{2}\ln 5g \Rightarrow x = -\frac{5}{2}\ln\left(1 - \frac{v^2}{5g}\right) \Rightarrow\ v^2 = 5g\left(1 - e^{-0,4x}\right);$

$g = 10 \Rightarrow v^2 = 50\left(1 - e^{-0,4x}\right)$.

Let $g(x) := 50\left(1 - e^{-0,4x}\right)$, then $v = \sqrt{g(x)}$ and

$\ddot{x} = \frac{dv}{dt} = \frac{d}{dt}\sqrt{g(x)} = \frac{1}{2}\cdot\frac{1}{\sqrt{g(x)}}\cdot\frac{dg(x)}{dx}\cdot\frac{dx}{dt}$.

But $\frac{dx}{dt} = v = \sqrt{g(x)} \Rightarrow \ddot{x} = \frac{1}{2}\cdot\frac{dg(x)}{dx} \Rightarrow\ \ddot{x} = 10e^{-0,4x}$.

8 Solution

Origin is point of release. $\downarrow$ is positive direction $\Rightarrow F = mg - mkv,\ k > 0$.

Equation of motion: $\ddot{x} = g - kv$.

Initial conditions: $t = 0,\ x = 0,\ v = 0$.

Terminal velocity V, but as $\ddot{x} \to 0,\ v \to \left(\frac{g}{k}\right)^{-}$. Hence $V = \frac{g}{k} \Rightarrow k = \frac{g}{V}$.

Expression relating v and x:

$$v\frac{dv}{dx} = g - kv,$$

$$dx = \frac{v}{g-kv}dv,$$

$$dx = -\frac{1}{k}\left(\frac{g-kv-g}{g-kv}\right)dv,$$

$$dx = \left(-\frac{1}{k} + \frac{g}{k^2}\cdot\frac{k}{g-kv}\right)dv,$$

$$x = -\frac{v}{k} + \frac{g}{k^2}\ln\frac{1}{g-kv} + c,\ c \text{ constant.}$$

$$x = 0,\ v = 0 \Rightarrow c = \frac{g}{k^2}\ln g;$$

$$x = -\frac{v}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right),$$

$$v = \frac{V}{2} \Rightarrow x = -\frac{V}{2k} + \frac{g}{k^2}\ln\left(\frac{1}{1-kV/2g}\right),$$

but $k = \frac{g}{V}$, hence

$$x = -\frac{V^2}{2g} + \frac{V^2}{g}\ln 2.$$

Expression relating v and t:

$$\frac{dv}{dt} = g - kv,$$

$$dt = \frac{dv}{g-kv},$$

$$dt = -\frac{1}{k}\cdot\frac{-k\,dv}{g-kv},$$

$$t = -\frac{1}{k}\ln(g-kv) + A,\ A$$

constant.

$$t = 0,\ v = 0 \Rightarrow A = \frac{1}{k}\ln g;$$

$$t = \frac{1}{k}\ln\left(\frac{g}{g-kv}\right),$$

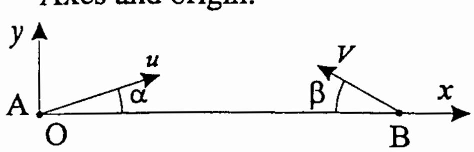
but $k = \frac{g}{V}$, hence

$$t = \frac{V}{g}\ln\left(\frac{1}{1-V\,v}\right),$$

$$v = \frac{V}{2} \Rightarrow t = \frac{V}{g}\ln 2.$$

9 Solution

Axes and origin:



$AB \equiv d = 110$,

$u = 60,\ \alpha = \frac{\pi}{6}$,

$V = 50,\ \beta = ?$

Initial conditions:

when $t = 0$

particle from A

particle from B

$x_1 = 0,\ y_1 = 0;$ $x_2 = d,\ y_2 = 0;$

$\dot{x}_1 = u\cos\alpha,\ \dot{y}_1 = u\sin\alpha;$

$\dot{x}_2 = -V\cos\beta,\ \dot{y}_2 = V\sin\beta.$

Hence after t seconds, the two particles are at positions:

$x_1 = u\cos\alpha\cdot t,$ $x_2 = d - V\cos\beta\cdot t,$

$y_1 = u\sin\alpha\cdot t - \frac{gt^2}{2},$ $y_2 = V\sin\beta\cdot t - \frac{gt^2}{2}.$

When the particles collide their coordinates are equal. Hence $x_1 = x_2$ and $y_1 = y_2$.

$$y_1 = y_2 \Rightarrow u\sin\alpha = V\sin\beta \Rightarrow \sin\beta = \frac{u}{V}\sin\alpha \Rightarrow$$

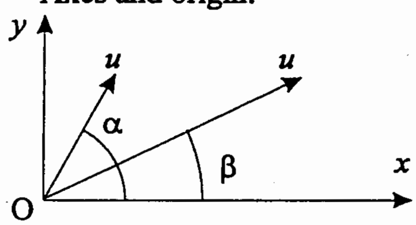
$$\sin\beta = \frac{6}{5}\sin\frac{\pi}{6} \Rightarrow \sin\beta = \frac{3}{5} \Rightarrow \cos\beta = \frac{4}{5}.\ \text{Hence } \tan\beta = \frac{3}{4} \Rightarrow \beta = \tan^{-1}\frac{3}{4}.$$

$$x_1 = x_2 \Rightarrow (u\cos\alpha + V\cos\beta)t = d \Rightarrow t = \frac{110}{60\cdot\frac{\sqrt{3}}{2} + 50\cdot\frac{4}{5}} \Rightarrow t = \frac{11}{3\sqrt{3}+4} \Rightarrow$$

$$t = \frac{11}{3\sqrt{3}+4}\cdot\frac{3\sqrt{3}-4}{3\sqrt{3}-4} \Rightarrow t = 3\sqrt{3} - 4s.$$

10 Solution

Axes and origin:



$u = 20;$

$\beta < \alpha,\ g = 10;$

$T = ?$

Let $t = 0$ be the time when the first particle is projected.

Initial conditions:

the first particle

$t = 0;$

$x_1 = 0,\ y_1 = 0;$

$\dot{x}_1 = u\cos\alpha,\ \dot{y}_1 = u\sin\alpha;$

the second particle

$t = T;$

$x_2 = 0,\ y_2 = 0;$

$\dot{x}_2 = u\cos\beta,\ \dot{y}_2 = u\sin\beta.$

Hence after t seconds, the two particles are at positions:

$x_1 = u\cos\alpha\cdot t,$ $x_2 = u\cos\beta\cdot(t - T),.$

$$y_1 = u\sin\alpha - \frac{gt^2}{2},$$

$$y_2 = u\sin\beta\cdot(t-T) - \frac{g(t-T)}{2}.$$

When the particles collide $x_1 = x_2 = 24$ and $y_1 = y_2 = 12$. Hence

$24 = 20\cos\alpha\cdot t$, $\quad 24 = 20\cos\beta\cdot(t-T)$,

$12 = 20\sin\alpha\cdot t - 5t^2$,

$12 = 20\sin\beta\cdot(t-T) - 5(t-T)^2$.

From here

$1{,}2 = \cos\alpha\cdot t$, (1) $\quad 1{,}2 = \cos\beta\cdot(t-T)$, (3)

$2{,}4 = 4\sin\alpha\cdot t - t^2$, (2) $\quad 2{,}4 = 4\sin\beta\cdot(t-T) - (t-T)^2$.

(4)

(1) $\Rightarrow t = \dfrac{1{,}2}{\cos\alpha} \Rightarrow$ from (2) $2{,}4 = 4\cdot 1{,}2\tan\alpha - \dfrac{1{,}44}{\cos^2\alpha}$.

But

$\dfrac{1}{\cos^2\alpha} = 1 + \tan^2\alpha \Rightarrow 2{,}4 = 4{,}8\tan\alpha - 1{,}44\left(1+\tan^2\alpha\right) \Rightarrow$

$1{,}44\tan^2\alpha - 4{,}8\tan\alpha + 3{,}84 = 0$;

$$\tan\alpha = \frac{2{,}4 \pm \sqrt{2{,}4^2 - 1{,}44\cdot 3{,}84}}{1{,}44};\ \tan\alpha = 2 \text{ or } \tan\alpha = \frac{4}{3}.$$

Analogously,

(3) $\Rightarrow t - T = \dfrac{1{,}2}{\cos\beta} \Rightarrow$ from (4) $2{,}4 = 4{,}8\tan\beta - \dfrac{1{,}2^2}{\cos^2\beta} \Rightarrow$

$1{,}2\tan^2\beta - 4\tan\beta + 3{,}2 = 0$;

$$\tan\beta = \frac{4 \pm \sqrt{16 - 4\cdot 1{,}2\cdot 3{,}2}}{2\cdot 1{,}2};\ \tan\beta = 2 \text{ or } \tan\beta = \frac{4}{3}.$$

We have $\alpha > \beta \Rightarrow \tan\alpha > \tan\beta \Rightarrow \tan\alpha = 2$ and $\tan\beta = \dfrac{4}{3}$. From here $\cos\alpha = \dfrac{1}{\sqrt{5}}$

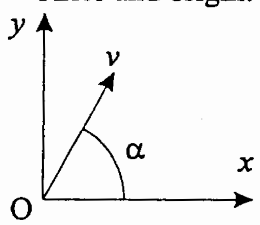
and $\cos\beta = \dfrac{3}{5}$.

From (1) $t = \dfrac{1{,}2}{\cos\alpha} \Rightarrow t = 1{,}2\sqrt{5}$.

From (3) $t-T=\dfrac{1{,}2}{\cos\beta}\Rightarrow T=1{,}2\sqrt{5}-\dfrac{1{,}2\cdot 5}{3}\Rightarrow\ T=1{,}2\sqrt{5}-2\Rightarrow\ T=\dfrac{6}{\sqrt{5}}-2.$

11 Solution

Axes and origin:



Initial conditions when $t=0$

$x=0,\ \ y=0;\ \dot{x}=v\cos\alpha,\ \ \dot{y}=v\sin\alpha.$

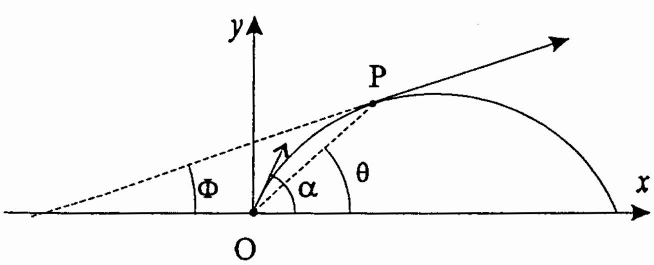
Equation of motion: $\ddot{x}=0,\ \ \ddot{y}=-g.$

Hence after t seconds we have

$$\dot{x}=v\cos\alpha, \qquad (1) \qquad\qquad x=v\cos\alpha\cdot t, \qquad (3)$$

$$\dot{y}=v\sin\alpha-gt, \qquad (2) \qquad\qquad y=v\sin\alpha\cdot t-\frac{gt^2}{2}. \qquad (4)$$

When the particle reaches its highest point $\dot{y}=0\Rightarrow$ from (2) the time of this event is $t=\dfrac{v\sin\alpha}{g}$. But we observe the particle at a time less than $\dfrac{v\sin\alpha}{g}$. Hence we have the following picture:



From the picture we see $\tan\theta=\dfrac{y}{x}\Rightarrow$ from (3) and (4)

$$\tan\theta=\tan\alpha-\frac{gt}{2v\cos\alpha}. \qquad (5)$$

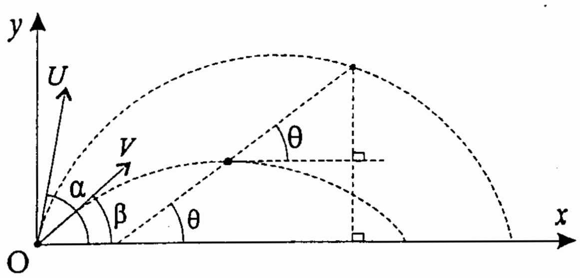
Analogously, $\tan\Phi=\dfrac{\dot{y}}{\dot{x}}\Rightarrow$ from (1) and (2)

$\tan\Phi=\tan\alpha-\dfrac{gt}{v\cos\alpha}$. Hence $\tan\Phi+\tan\alpha=2\left(\tan\alpha-\dfrac{gt}{2v\cos\alpha}\right)\ \Rightarrow$ from (5)

$\tan\Phi+\tan\alpha=2\tan\theta.$

12 Solution

Axes, origin and dimension diagram



Initial conditions when $t=0$

$x_1=0,\ y_1=0;$ $\qquad x_2=0,\ y_2=0;$

$\dot{x}_1=U\cos\alpha;$ $\qquad \dot{x}_2=V\cos\beta;$

$\dot{y}_1=U\sin\alpha;$ $\qquad \dot{y}_2=V\sin\beta.$

After t seconds the particles are at positions.

$$x_1=U\cos\alpha\cdot t,$$

$$y_1=U\sin\alpha\cdot t-\frac{gt^2}{2},$$

and

$$x_2=V\cos\beta\cdot t,$$

$$y_2=V\sin\beta\cdot t-\frac{gt^2}{2}.$$

As seen from the picture $\tan\theta=\dfrac{y_2-y_1}{x_2-x_1}\Rightarrow\tan\theta=\dfrac{V\sin\beta-U\sin\alpha}{V\cos\beta-U\cos\alpha}.$

13 Solution

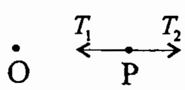
(a) The resultant force is $\dfrac{mv^2}{l}$ to A where $l=2\,\text{m}$. Let T be the tension in the string. Hence

$$T=\frac{mv^2}{l}\Rightarrow T=\frac{0{,}5\cdot 12^2}{2}=36\,\text{N}.$$

(b) $T=64\Rightarrow\dfrac{mv^2}{l}=64\Rightarrow\ l=\left(\dfrac{64\cdot l}{m}\right)^{1/2}\Rightarrow\ v=\left(\dfrac{64\cdot 2}{0{,}5}\right)^{1/2}=16\,\text{m}s^{-1}.$

14 Solution

Forces on the particle



where $T_1=4v,\ T_2=\dfrac{k}{r},\ k>0.$

(a) If $t=\dfrac{\pi}{10}$ is the time of one revolution, then the angular velocity

$$\omega=\frac{2\pi}{t}\Rightarrow\ \omega=20\,rad\,s^{-1}.$$

$v = \omega \cdot r$ and $v = 40 \Rightarrow r = \dfrac{v}{\omega} \Rightarrow r = \dfrac{40}{20} = 2\,\text{m}$.

The resultant force on the particle is $T_1 - T_2 = 4v - \dfrac{k}{r}$ to O, hence $4v - \dfrac{k}{r} = \dfrac{mv^2}{r} \Rightarrow$

$k = 4vr - mv^2$;

$v = 40,\ r = 2,\ m = 0{,}1 \Rightarrow k = 4 \cdot 40 \cdot 2 - 0{,}1 \cdot 40^2 \Rightarrow k = 160\,\text{N}$.

(b) From (a) $4v - \dfrac{k}{r} = \dfrac{mv^2}{r} \Rightarrow\ v^2 - \dfrac{4r}{m}v + \dfrac{k}{m} = 0$;

$k = 30,\ r = 1,\ m = 0{,}1 \Rightarrow\ v^2 - 40v + 300 = 0,\ v = 20 \pm \sqrt{400 - 300}$; $v = 30$ or

$v = 10\ \text{m}\,s^{-1}$.

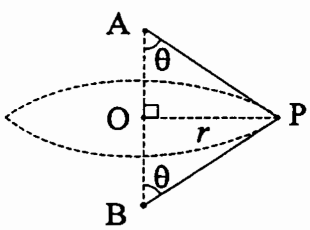
(c) If the particle describes a circle its velocity $v > 0$. Find the values of v in accordance with k. From (a) $4v - \dfrac{k}{r} = \dfrac{mv^2}{r} \Rightarrow\ v^2 - \dfrac{4r}{m}v + \dfrac{k}{m} = 0$;

$r = 1,\ m = 0{,}1 \Rightarrow\ v^2 - 40v + 10k = 0 \Rightarrow\ v = 20 \pm \sqrt{400 - 10k}$.

Hence the values of v are positive if and only if $400 - 10k \geq 0 \Rightarrow k \leq 40$. So we have $0 < k \leq 40$.

15 Solution

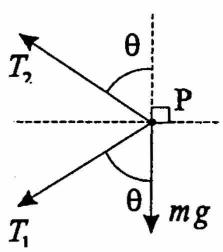
Dimension diagram



AP=BP=$\dfrac{5}{2}l$,

AO=BO=$2l$.

Forces on P



The resultant force on P is horizontal towards O of magnitude $\dfrac{mv^2}{r}$.

(a) The vertical component is zero $\Rightarrow T_2\cos\theta - T_1\cos\theta = mg$. (1)

The horizontal component is $\dfrac{mv^2}{r}$ $\Rightarrow T_2\sin\theta + T_1\sin\theta = \dfrac{mv^2}{r}$. (2)

But $\cos\theta = \dfrac{\text{AO}}{\text{AP}} \Rightarrow \cos\theta = \dfrac{4}{5} \Rightarrow \sin\theta = \dfrac{3}{5}$, and $r = \sqrt{\text{AP}^2 - \text{AO}^2} \Rightarrow r = \dfrac{3}{2}l$.

Hence from (1) $T_2 - T_1 = \frac{5}{4} m g$, (3)

and from (2) $T_2 + T_1 = \frac{10}{9} \cdot \frac{m v^2}{l}$. (4)

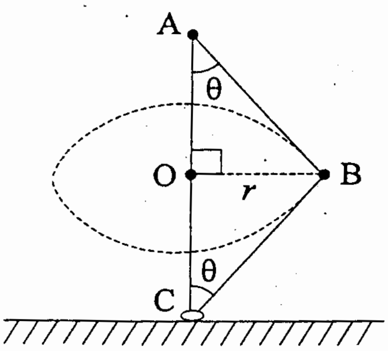
$$(3) + (4) \Rightarrow T_2 = \frac{5}{8} m g + \frac{5}{9} \frac{m v^2}{l}; \quad (4) - (3) \Rightarrow T_1 = \frac{5}{9} \frac{m v^2}{l} - \frac{5}{8} m g.$$

(b) The motion described in the problem is possible if $T_1 > 0$, as the strings are both taut.

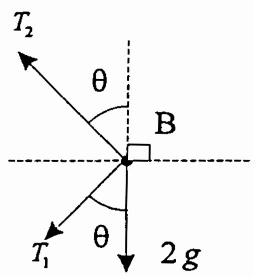
$T_1 > 0 \Rightarrow$ from (a) $\frac{5}{9} \frac{m v^2}{l} - \frac{5}{8} m g > 0 \Rightarrow 8v^2 > 9gl$.

16 Solution

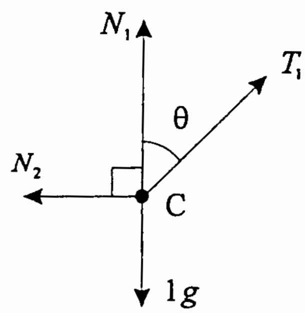
Dimension diagram



Forces on B



Forces on C



AB=BC=$\frac{1}{2}$, AC=$\frac{\sqrt{3}}{2}$.

N_2 is the force exerted by the rod AC on the ring C, and N_1 is the force exerted by the ledge.

(a) The resultant force on B is $2\omega^2 r$ towards O.

The vertical component is zero $\Rightarrow T_2 \cos\theta - T_1 \cos\theta = 2g$. (1)

The horizontal component is $2\omega^2 r \Rightarrow T_2 \sin\theta + T_1 \sin\theta = 2\omega^2 r$. (2)

But $\omega = 6$, $r = \sqrt{\mathrm{AB}^2 - \mathrm{AO}^2} \Rightarrow \quad r = \frac{1}{4}$, $\cos\theta = \frac{\mathrm{AO}}{\mathrm{AB}} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{1}{2}$.

Hence from (1) and.(2) we obtain:

$$T_2 - T_1 = \frac{4g}{\sqrt{3}}, \quad (3)$$

$$T_2 + T_1 = 36. \quad (4)$$

$$(3) + (4) \Rightarrow T_2 = \frac{2g}{\sqrt{3}} + 18 \Rightarrow T_2 = \frac{20}{\sqrt{3}}\text{N} + 18\,;$$

$$(4) - (3) \Rightarrow T_1 = 18 - \frac{2g}{\sqrt{3}} \Rightarrow T_1 = 18 - \frac{20}{\sqrt{3}}\text{N}\,.$$

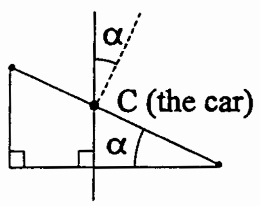
(b) The resultant force on C is zero. For its vertical component we have

$$N_1 + T_1\cos\theta = 1g \Rightarrow N_1 = g - \left(18 - \frac{20}{\sqrt{3}}\right)\frac{\sqrt{3}}{2} \Rightarrow$$

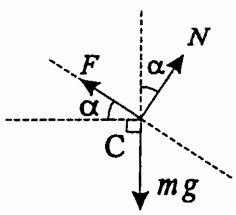
$$N_1 = g + 10 - 9\sqrt{3} \Rightarrow N_1 = 20 - 9\sqrt{3}\,\text{N}\,.$$

17 Solution

Dimension diagram



Forces on C



Here F is a friction force on C from the surface. N is a reaction force. If the car has no tendency to slip sideways, then F must equal zero. Find the value of F.

The resultant force on C is $\dfrac{mv^2}{r}$ horizontally to the left. The vertical component is zero $\Rightarrow F\sin\alpha + N\cos\alpha = mg$. (1)

The horizontal component is $\dfrac{mv^2}{r}$ $\Rightarrow -F\cos\alpha + N\sin\alpha = \dfrac{mv^2}{r}$. (2)

$$(1)\times\sin\alpha - (2)\times\cos\alpha \Rightarrow F\cdot\left(\sin^2\alpha + \cos^2\alpha\right) = mg\sin\alpha - \frac{mv^2}{r}\cos\alpha \Rightarrow$$

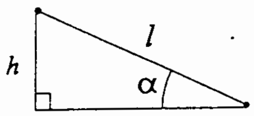
$$F = m\cos\alpha\left(g\tan\alpha - \frac{v^2}{r}\right);$$

$$v = 60\,kmh^{-1} = \frac{100}{6}ms^{-1},\ r = 100\,\text{m} \Rightarrow\ g\cdot\tan\alpha - \frac{v^2}{r} = 10\cdot\frac{5}{18} - \frac{10^4}{36\cdot 100} = 0,$$ and

hence $F = 0$.

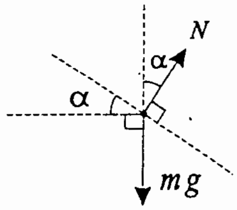
18 Solution

Dimension diagram



$h = 0{,}1;\ l = 1{,}5;\ r = 600;$

Forces on the wheels



If a sideways trust is eliminated, then the only force on the wheels is a reaction force N at right angle to the surface of the rails.

The resultant force on the wheels is $\dfrac{mv^2}{r}$ horizontally to the right.

The vertical component is zero $\Rightarrow N\cos\alpha = mg \Rightarrow\ N = \dfrac{mg}{\cos\alpha}$. (1)

The horizontal component is $\dfrac{mv^2}{r}$ $\Rightarrow N\sin\alpha = \dfrac{mv^2}{r} \Rightarrow\ v^2 = \dfrac{rN\sin\alpha}{m}$ and from (1)

$v^2 = rg\tan\alpha$. But $\tan\alpha = \dfrac{h}{l} \Rightarrow \tan\alpha = \dfrac{0{,}1}{1{,}5} = \dfrac{1}{15}$. Hence

$$v = \left(600 \cdot g \cdot \frac{1}{15}\right)^{1/2} \Rightarrow v = 20\,ms^{-1} \text{ if } g = 10 \text{ and } v = 19{,}8\,ms^{-1} \text{ if } g = 9{,}81.$$

Further Questions 7

1 Solution

Choose initial position as origin, and initial direction as positive.

Initial conditions: $t=0,\ x=0,\ v=1$.

Equation of motion: $\ddot{x}=-e^{v}$.

(a) Relation between v and t: $\dfrac{dv}{dt}=-e^{v}\Rightarrow dt=-e^{v}dv\Rightarrow t=-e^{v}+c,\ c$ constant.

$t=0,\ v=1\Rightarrow\ c=-e^{-1}\Rightarrow\ t=e^{-v}-e^{-1};\quad v=\dfrac{1}{2},\ t=t_1\Rightarrow t_1=e^{-1/2}-e^{-1};$

$v=0,\ t=t_2+t_1\Rightarrow t_2+t_1=1-e^{-1}\Rightarrow\ t_2=\left(1-e^{-1}\right)-\left(e^{-1/2}-e^{-1}\right)\Rightarrow t_2=1-e^{-1/2}\Rightarrow$

$$\frac{t_2}{t_1}=\frac{1-e^{-1/2}}{e^{-1/2}-e^{-1}}=\frac{1-e^{-1/2}}{e^{-1/2}\left(1-e^{-1/2}\right)}\Rightarrow\ \frac{t_2}{t_1}=e^{1/2}.$$

(b) Relation between v and x: $\dfrac{v\,dv}{dx}=-e^{v}\Rightarrow dx=-v\,e^{-v}dv\Rightarrow x=v\,e^{-v}+e^{-v}+c,\ c$ constant.

$x=0,\ v=1\Rightarrow c=-2e^{-1}\Rightarrow x=e^{-v}(v+1)-2e^{-1}:\ v=0\Rightarrow x=1-2e^{-1}.$

2 Solution

Choose initial direction as positive.

Initial conditions: $t=0,\ x=1,\ v=0$.

Equation of motion: $\ddot{x}=\dfrac{k}{x^2},\ k>0$.

(a) Relation between v and x: $\dfrac{v\,dv}{dx}=\dfrac{k}{x^2}\Rightarrow vdv=\dfrac{k}{x^2}dx\Rightarrow\dfrac{v^2}{2}=-\dfrac{k}{x}+c,\ c$ constant.

$$x=1,\ v=0\Rightarrow c=k\Rightarrow v^2=2k\left(1-\frac{1}{x}\right); \qquad (1)$$

$x\geq 1\Rightarrow 1-\dfrac{1}{x}<1\Rightarrow v^2<2k\Rightarrow\ v<\sqrt{2k}$.

(b) Relation between v and t : $\frac{dv}{dt}=\frac{k}{x^2}$, but from (1) $\frac{1}{x}=1-\frac{v^2}{2k}\Rightarrow$

$\frac{dv}{dt}=\frac{\left(2k-v^2\right)^2}{4k}$. From here $dt=\frac{4k\,dv}{\left(2k-v^2\right)^2}\Rightarrow t+c=\int\frac{4k\,dv}{\left(2k-v^2\right)^2}$, c constant.

$$f(v):=\int\frac{4k\,dv}{\left(2k-v^2\right)^2}=2\int\frac{\left(2k-v^2+v^2\right)dv}{\left(2k-v^2\right)^2}=2\left\{\int\frac{dv}{2k-v^2}+\int\frac{v^2\,dv}{\left(2k-v^2\right)^2}\right\}=$$

$$2\int\frac{dv}{2k-v^2}+2\int\frac{v\,d\left(2k-v^2\right)}{(-2)\left(2k-v^2\right)^2}=2\int\frac{dv}{2k-v^2}-\left\{\frac{-v}{\left(2k-v^2\right)}+\int\frac{dv}{2k-v^2}\right\}=$$

$$\int\frac{dv}{2k-v^2}+\frac{v}{2k-v^2}=\frac{1}{2\sqrt{2}}\int dv\left(\frac{1}{\sqrt{2k}-v}+\frac{1}{\sqrt{2k}+v}\right)+\frac{v}{2k-v^2}=$$

$$\frac{1}{2\sqrt{2k}}\ln\frac{\sqrt{2k}+v}{\sqrt{2k}-v}+\frac{v}{2k-v^2}.$$

Let at time $t=t_1$, $v=v_1$ and $x=2$, and let at time $t=t_2$, $v=v_2$ and $x=4$.

$x=2\Rightarrow$ from (1) $v_1=\sqrt{k}$;

$x=4\Rightarrow$ from (1) $v_2=\sqrt{\frac{3}{2}k}$.

We obtained that $t+c=f(v)$, where c is constant and

$$f(v)=\frac{1}{2\sqrt{2k}}\ln\frac{\sqrt{2k}+v}{\sqrt{2k}-v}+\frac{v}{2k-v^2}.$$

Hence $t_2-t_1=f(v_2)-f(v_1)\Rightarrow$

$$t_2-t_1=\frac{1}{\sqrt{2k}}\left\{\frac{1}{2}\ln\frac{\sqrt{2k}+\sqrt{\frac{3}{2}k}}{\sqrt{2k}-\sqrt{\frac{3}{2}k}}-\frac{1}{2}\ln\frac{\sqrt{2k}+\sqrt{k}}{\sqrt{2k}-\sqrt{k}}+\sqrt{2k}\cdot\frac{\sqrt{\frac{3}{2}k}}{2k-\frac{3}{2}k}-\sqrt{2k}\cdot\frac{\sqrt{k}}{2k-k}\right\}\Rightarrow$$

$t_2-t_1=\frac{1}{\sqrt{2k}}\left\{\frac{1}{2}\ln\frac{2+\sqrt{3}}{2-\sqrt{3}}+\frac{1}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}+2\sqrt{3}-\sqrt{2}\right\}$, but $\frac{1}{2-\sqrt{3}}=2+\sqrt{3}$ and

$\frac{1}{\sqrt{2}-1}=\sqrt{2}+1$,

hence $t_2 - t_1 = \dfrac{1}{2k}\left\{\ln\dfrac{2+\sqrt{3}}{2+\sqrt{2}} + 2\sqrt{3} - \sqrt{2}\right\}$.

3 Solution

Motion is simple harmonic $\Rightarrow$ $x = a\cos(nt+\alpha)$, $0 \le \alpha < 2\pi$, $\Rightarrow \dot{x} = -an\sin(nt+\alpha)$.

$t = t_1$, $x = x_1$, $v = v_1$; $\qquad$ $t = t_2$, $x = x_2$, $v = v_2$;

$\Rightarrow x_1 = a\cos(nt_1+\alpha)$; (1) $\qquad$ $\Rightarrow x_2 = a\cos(nt_2+\alpha)$; (3)

$\Rightarrow y_1 = -an\sin(nt_1+\alpha)$; (2) $\qquad$ $\Rightarrow y_2 = -an\sin(nt_2+\alpha)$. (4)

From (1) and (2) $x_1^2 + \dfrac{v_1^2}{n^2} = a^2$, and from (3) and (4) $x_2^2 + \dfrac{v_2^2}{n^2} = a^2$. Hence

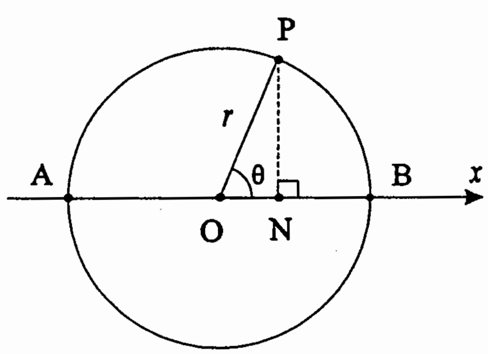
$$x_1^2 + \frac{v_1^2}{n^2} = x_2^2 + \frac{v_2^2}{n^2} \Rightarrow n^2\left(x_2^2 - x_1^2\right) = v_1^2 - v_2^2 \Rightarrow n = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}.$$

But $T = \dfrac{2\pi}{n} \Rightarrow T = 2\pi \cdot \sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$.

$$a^2 = x_1^2 + \frac{v_1^2}{n^2} \Rightarrow a^2 = x_1^2 + \frac{v_1^2\left(x_2^2 - x_1^2\right)}{v_1^2 - v_2^2} \Rightarrow a^2 = \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2} \Rightarrow a = \sqrt{\frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}}.$$

4 Solution

Dimension diagram



Choose the center of a circle as origin. If x is the coordinate of P, then $x = r\cos\theta$. But $\theta = \omega t \Rightarrow x = r\cos\omega t$, and hence the motion of N is simple harmonic.

5 Solution

(a) Upward motion. Choose a point of projection as origin and ↑ as positive.

Initial conditions: $t = 0$, $x = 0$, $v = 2c$.

Equation of motion: $\ddot{x} = -g - \frac{g}{c^2}v^2$.

Expression relating x and v.

$$v\frac{dv}{dx} = -g - \frac{g}{c^2}v^2,$$

$$-g dx = \frac{v\,dv}{1+\frac{v^2}{c^2}},$$

$$-gx + A = \frac{c^2}{2}\ln\left(1+\frac{v^2}{c^2}\right),\ A \text{ constant;}$$

$$x=0,\ v=2c \Rightarrow A = \frac{c^2}{2}\ln 5 \Rightarrow$$

$$x = \frac{c^2}{2g}\ln\frac{5c^2}{c^2+v^2}. \qquad (1)$$

Expression relating v and t.

$$\frac{dv}{dt} = -g - \frac{g}{c^2}v^2,$$

$$-g\,dt = \frac{dv}{1+\frac{v^2}{c^2}},$$

$$-gt + A = c\cdot\tan^{-1}\frac{v}{c},\ A \text{ constant;}$$

$$t=0,\ v=2c \Rightarrow A = c\cdot\tan^{-1}2 \Rightarrow$$

$$t = \frac{c}{g}\left(\tan^{-1}2 - \tan^{-1}v\right). \qquad (2)$$

When the particle reaches its highest point, its velocity is zero. So $v = 0 \Rightarrow$ from (2)

$t = \dfrac{c\cdot\tan^{-1}2}{g}$ is the time of ascent.

(b) Let h be the distance between the point of projection and the highest point. Then $v=0 \Rightarrow$ from (1) $h = \dfrac{c^2}{2g}\ln 5$. Downward motion. Origin at highest point and $\downarrow$ as positive direction.

Initial conditions: $t=0,\ x=0,\ v=0$.

Equation of motion: $\ddot{x} = g - \frac{g}{c^2}v^2$.

Terminal velocity: as $\ddot{x} \to 0,\ v \to (c)^- \Rightarrow v < c$.

Expression relating x and v: $v\dfrac{dv}{dx} = g - \dfrac{g}{c^2}v^2$

$$\Rightarrow g dx = \frac{v\,dv}{1-\frac{v^2}{c^2}} \Rightarrow gx + A = \frac{-c^2}{2}\ln\left(1-\frac{v^2}{c^2}\right),\ A \text{ constant; } x=0,\ v=0 \Rightarrow A=0 \Rightarrow$$

$$x = \frac{c^2}{2g}\ln\frac{c^2}{c^2-v^2}. \qquad (3)$$

When the particle returns to its starting point, $x = h$. Hence from (3)

$h = \frac{c^2}{2g}\ln\frac{c^2}{c^2 - v^2}$. But $h = \frac{c^2}{2g}\ln 5 \Rightarrow 5 = \frac{c^2}{c^2 - v^2} \Rightarrow v = \frac{2c}{\sqrt{5}}$.

6 Solution

Upward motion. Choose a point of projection as origin and $\uparrow$ as positive.

Initial conditions: $t = 0,\ x = 0,\ v = nV$.

Equation of motion: $\ddot{x} = -g - kv^2$.

Expression relating x and v.

$$v\frac{dv}{dx} = -g - kv^2,$$

$$-dx = \frac{v\,dv}{g + kv^2},$$

$$-dx = \frac{1}{2k}\cdot\frac{2kv\,dv}{g + kv^2},$$

$$-x + c = \frac{1}{2k}\ln\left(g + kv^2\right),\ c \text{ constant;}$$

Expression relating v and t.

$$\frac{dv}{dt} = -g - kv^2,$$

$$-dt = \frac{dv}{g + kv^2},$$

$$-dt = \frac{1}{g}\frac{dv}{1 + \left(\sqrt{\frac{k}{g}}\,v\right)^2},$$

$$-t + A = \frac{1}{\sqrt{gk}}\cdot\tan^{-1}\left(\sqrt{\frac{k}{g}}\,v\right),\ A$$

constant;

$$x = 0,\ v = nV \Rightarrow c = \frac{1}{2k}\ln\left(g + kn^2V^2\right)$$

$$t = 0,\ v = nV \Rightarrow A = \frac{1}{\sqrt{gk}}\cdot\tan^{-1}\left(\sqrt{\frac{k}{g}}\,nV\right)$$

$$\Rightarrow x = \frac{1}{2k}\ln\left(\frac{g + kn^2V^2}{g + kv^2}\right). \qquad (1)$$

$$\Rightarrow t = \frac{1}{\sqrt{gk}}\left\{\tan^{-1}\sqrt{\frac{k}{g}}\,nV - \tan^{-1}\sqrt{\frac{k}{g}}\,v\right\}. \qquad (2)$$

Let h be the distance between the point of projection and the highest point and t_1 be the time of ascent. When the particle reaches its highest point, its velocity is zero.

Then $v = 0 \Rightarrow$ from (1) $h = \frac{1}{2k}\ln\left(1 + \frac{k}{g}n^2V^2\right)$, and from (2) $t_1 = \frac{1}{\sqrt{gk}}\tan^{-1}\sqrt{\frac{k}{g}}\,nV$.

Downward motion. Choose the highest point as origin and $\downarrow$ as positive.

Initial conditions: $t=0,\ x=0,\ v=0$.

Equation of motion: $\ddot{x}=g-k\,v^2$.

Terminal velocity: as $\ddot{x}\to 0,\ v^2\to\left(\frac{g}{k}\right)$. Hence $V^2=\frac{g}{k}\Rightarrow k=\frac{g}{V^2}$. Using this equation, $h=\frac{V^2}{2g}\ln\left(1+n^2\right)$ and $t_1=\frac{V}{g}\tan^{-1}n$.

Expression relating x and v.

$$v\frac{dv}{dx}=g-k\,v^2,$$

$$dx=\frac{v\,dv}{g-k\,v^2},$$

$$dx=-\frac{1}{2k}\cdot\frac{-2kv\,dv}{g-k\,v^2},$$

$$x+c=\frac{-1}{2k}\ln\left(g-k\,v^2\right),\ c\ \text{constant};$$

$$x=0,\ v=0\Rightarrow c=\frac{-1}{2k}\ln g,$$

$$\Rightarrow x=\frac{1}{2k}\ln\left(\frac{g}{g-k\,v^2}\right). \qquad (3)$$

Expression relating v and t.

$$\frac{dv}{dt}=g-k\,v^2,$$

$$dt=\frac{dv}{g-k\,v^2},$$

$$dt=\frac{1}{2\sqrt{gk}}\left(\frac{1}{\sqrt{g}-\sqrt{k}\,v}+\frac{1}{\sqrt{g}+\sqrt{k}\,v}\right)\sqrt{k}\,dv,$$

$$t+A=\frac{1}{2\sqrt{g\,k}}\cdot\ln\left(\frac{\sqrt{g}+\sqrt{k}\,v}{\sqrt{g}-\sqrt{k}\,v}\right),\ A$$

constant;

$$t=0,\ v=0\Rightarrow A=0,$$

$$\Rightarrow t=\frac{1}{2\sqrt{g\,k}}\ln\left(\frac{1+\sqrt{\frac{k}{g}}\,v}{1-\sqrt{\frac{k}{g}}\,v}\right). \qquad (4)$$

Let v_1 be the speed with which the particle returns to its starting point and t_2 be the time taken by this particle to return from its highest point to its starting point. Then $x=h,\ v=v_1\Rightarrow$ from (3)

$$h = \frac{1}{2k}\ln\left(\frac{g}{g-kv_1^2}\right) \Rightarrow \frac{1}{2k}\ln\left(1+\frac{k}{g}n^2V^2\right) = \frac{1}{2k}\ln\left(\frac{1}{1-\frac{k}{g}v_1^2}\right).$$ Using $k = \frac{g}{V^2}$,

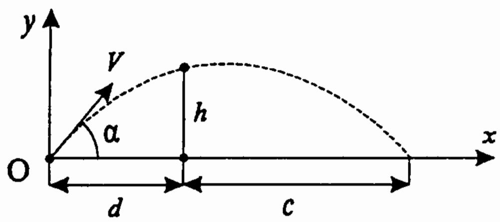
$1+n^2 = \dfrac{1}{1-\dfrac{v_1^2}{V^2}} \Rightarrow v_1 = V\cdot\sqrt{\dfrac{n^2}{n^2+1}}$; $v = v_1, t = t_2 \Rightarrow$ from (4) $t_2 = \dfrac{1}{2\sqrt{gk}}\ln\left(\dfrac{1+\dfrac{v_1}{V}}{1-\dfrac{v_1}{V}}\right) \Rightarrow$

$t_2 = \dfrac{V}{2g}\ln\left(\dfrac{\sqrt{n^2+1}+\sqrt{n^2}}{\sqrt{n^2+1}-\sqrt{n^2}}\right)$. But $\dfrac{1}{\sqrt{n^2+1}-\sqrt{n^2}} = \sqrt{n^2+1}+\sqrt{n^2}$, hence

$t_2 = \dfrac{V}{g}\ln\left(n+\sqrt{n^2+1}\right)$. From here $t_1 + t_2 = \dfrac{V}{g}\left\{\tan^{-1}n + \ln\left(n+\sqrt{n^2+1}\right)\right\}$ is the time taken by the particle to return to its starting point.

7 Solution

Axes and origin



Initial conditions when $t = 0$

$$x = 0,\ y = 0;$$

$$\dot{x} = V\cos\alpha,\ \dot{y} = V\sin\alpha.$$

Equation of motion

$$x = V\cos\alpha\cdot t \quad (1),\quad y = V\sin\alpha\cdot t - \frac{gt^2}{2}.$$

(2)

(a) $x = d \Rightarrow$ from (1) $t = \dfrac{d}{V\cos\alpha}$, then $y = h \Rightarrow$ from (2)

$$h = V\sin\alpha\cdot\frac{d}{V\cos\alpha} - \frac{g}{2}\cdot\frac{d^2}{V^2\cos^2\alpha} \Rightarrow h = d\tan\alpha - \frac{g}{2}\cdot\frac{d^2}{\cos^2\alpha}\cdot\frac{1}{V^2} \Rightarrow$$

$$V^2 = \frac{gd^2}{2\cos^2\alpha}\cdot\frac{1}{d\tan\alpha - h}. \qquad (3)$$

Let T be the time when the particles hits the ground. Then $y = 0 \Rightarrow$ from (2)

$$0 = V\sin\alpha - \frac{gT}{2} \Rightarrow T = \frac{2V\sin\alpha}{g}.$$

$t = T \Rightarrow$ from (1) $c + d = V\cos\alpha \cdot T \Rightarrow c + d = \dfrac{2V^2\cos\alpha\sin\alpha}{g}$. Hence using (3),

$$c = \frac{2\cos\alpha\sin\alpha}{g}\cdot\frac{gd^2}{2\cos^2\alpha}\cdot\frac{1}{(d\tan\alpha - h)} - d \Rightarrow c = \frac{d^2\tan\alpha}{d\tan\alpha - h} - d \Rightarrow c = \frac{dh}{d\tan\alpha - h}.$$

(b) From (a) $c = \dfrac{2V^2}{g}\cos\alpha\sin\alpha - d$, $h = d\tan\alpha - \dfrac{gd^2}{2V^2}\cdot\dfrac{1}{\cos^2\alpha}$.

But $\cos\alpha\cdot\sin\alpha = \dfrac{\cos\alpha\cdot\sin\alpha}{\cos^2\alpha + \sin^2\alpha} = \dfrac{\tan\alpha}{1+\tan^2\alpha}$, and $\dfrac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$. Hence

$c + d = \dfrac{2V^2}{g}\cdot\dfrac{\tan\alpha}{1+\tan^2\alpha}$, $h = d\tan\alpha - \dfrac{gd^2}{2V^2}(1+\tan^2\alpha)$. From here, obviously,

$$\frac{\tan\alpha}{1+\tan^2\alpha} = \frac{g}{2V^2}(c+d), \tag{4}$$

$$\frac{h}{1+\tan^2\alpha} = d\frac{\tan\alpha}{1+\tan^2\alpha} - \frac{gd^2}{2V^2}. \tag{5}$$

Substituting (4) into (5), $\dfrac{h}{1+\tan^2\alpha} = \dfrac{dg}{2V^2}(c+d) - \dfrac{gd^2}{2V^2} \Rightarrow \dfrac{h}{1+\tan^2\alpha} = \dfrac{dgc}{2V^2} \Rightarrow$

$1+\tan^2\alpha = \dfrac{2hV^2}{dgc}$, and hence $\tan\alpha = \sqrt{\dfrac{2hV^2}{dgc} - 1}$. Substituting these expressions for

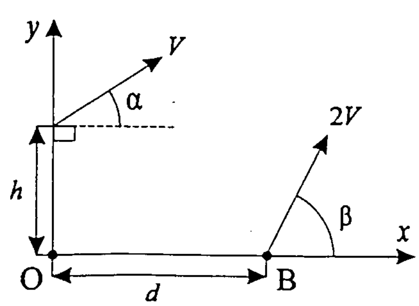
$1+\tan^2\alpha$ and $\tan\alpha$ into (4), $(c+d)\dfrac{h}{dc} = \sqrt{\dfrac{2hV^2}{dgc} - 1} \Rightarrow$

$$(c+d)^2h^2 = d^2c^2\left\{\frac{2hV^2}{dgc} - 1\right\} \Rightarrow$$

$$g\left\{d^2c^2 + (c+d)^2h^2\right\} = 2dhV^2c.$$

8 Solution

Axes and origin



Initial conditions when $t = 0$

$x_1 = 0,\ y_1 = h$; $\qquad x_2 = d,\ y_2 = 0$;

$\dot{x}_1 = V\cos\alpha$; $\qquad \dot{x}_2 = 2V\cos\beta$;

$\dot{y}_1 = V\sin\alpha$; $\qquad \dot{y}_2 = 2V\sin\beta$;

Equation of motion

$$x_1 = V\cos\alpha \cdot t, \qquad (1) \qquad\qquad x_2 = d + 2V\cos\beta \cdot t, \quad (3)$$

$$y_1 = h + V\sin\alpha \cdot t - \frac{gt^2}{2}, \qquad (2) \qquad\qquad y_2 = 2V\sin\beta \cdot t - \frac{gt^2}{2}. \ (4)$$

When the particles collide, their coordinates are equal. Hence

$x_1 = x_2 \Rightarrow$ from (1) and (3) $V\cos\alpha \cdot t = d + 2V\cos\beta \cdot t \Rightarrow t = \dfrac{d}{V(\cos\alpha - 2\cos\beta)}$.

$y_1 = y_2 \Rightarrow$ from (2) and (4) $h + V\sin\alpha \cdot t = 2V\sin\beta \cdot t \Rightarrow t = \dfrac{h}{V(2\sin\beta - \sin\alpha)}$.

Equating the two expressions for the time of collision t,

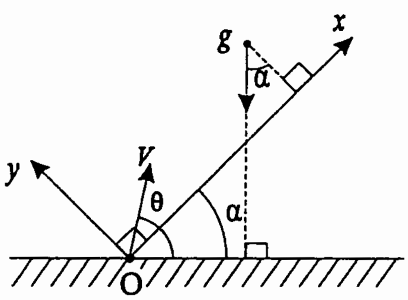
$\dfrac{d}{\cos\alpha - 2\cos\beta} = \dfrac{h}{2\sin\beta - \sin\alpha}$. But $\dfrac{h}{d} = \tan\gamma \Rightarrow$

$(2\sin\beta - \sin\alpha)\cos\gamma = \sin\gamma(\cos\alpha - 2\cos\beta) \Rightarrow$

$2(\sin\beta\cos\gamma + \cos\beta\sin\gamma) = \sin\alpha\cos\gamma + \cos\alpha\sin\gamma \Rightarrow \ 2\sin(\beta+\gamma) = \sin(\alpha+\gamma)$.

9 Solution

Axes and origin



Initial conditions when $t = 0$

$x = 0, \ y = 0;$

$\dot{x} = V\cos(\theta - \alpha);$

$\dot{y} = V\sin(\theta - \alpha).$

Equations of motion:

$\ddot{x} = -g\sin\alpha,$

$\dot{x} = V\cos(\theta - \alpha) - g\sin\alpha \cdot t,.$

$x = V\cos(\theta - \alpha) \cdot t - \dfrac{g\sin\alpha}{2} \cdot t^2. \qquad (1)$

$\ddot{y} = -g\cos\alpha,$

$\dot{y} = V\sin(\theta - \alpha) - g\cos\alpha \cdot t,$

$y = V\sin(\theta - \alpha) \cdot t - \dfrac{g\cos\alpha}{2} \cdot t^2. \quad (2)$

(a) Let T be the time when the particle reaches the inclined plane and R be the range on this plane. Then $y=0 \Rightarrow$ from (2) $V\sin(\theta-\alpha)-\frac{g\cos\alpha}{2}\cdot T=0 \Rightarrow$

$$T=\frac{2V}{g}\frac{\sin(\theta-\alpha)}{\cos\alpha}.$$

$$t=T \Rightarrow \text{ from (1) } R=V\cos(\theta-\alpha)\cdot\frac{2V}{g}\cdot\frac{\sin(\theta-\alpha)}{\cos\alpha}-\frac{g}{2}\sin\alpha\left(\frac{2V}{g}\right)^2\cdot\frac{\sin^2(\theta-\alpha)}{\cos^2\alpha} \Rightarrow$$

$$R=\frac{2V^2\sin(\theta-\alpha)}{g\cos^2\alpha}\{\cos(\theta-\alpha)\cos\alpha-\sin(\theta-\alpha)\sin\alpha\} \Rightarrow$$

$$R=\frac{2V^2\sin(\theta-\alpha)}{g\cos^2\alpha}\cdot\cos(\theta-\alpha+\alpha) \Rightarrow R=\frac{2V^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}.$$

(b) From (a) $R=\frac{2V^2}{g\cos^2\alpha}\cos\theta\sin(\theta-\alpha)$. (3)

Let $f(\theta):=\cos\theta\sin(\theta-\alpha)$. Hence $f(\theta)=\cos\theta(\sin\theta\cos\alpha-\cos\theta\sin\alpha) \Rightarrow$

$$f(\theta)=\frac{\sin2\theta\cos\alpha}{2}-\frac{\cos^2\theta\sin\alpha}{2} \Rightarrow f(\theta)=\frac{\sin2\theta\cos\alpha}{2}-\frac{(1+\cos2\theta)\sin\alpha}{2} \Rightarrow$$

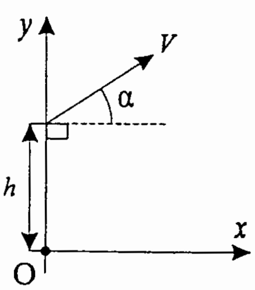
$$f(\theta)=\frac{\sin2\theta\cos\alpha-\cos2\theta\sin\alpha}{2}-\frac{\sin\alpha}{2} \Rightarrow f(\theta)=\frac{\sin(2\theta-\alpha)-\sin\alpha}{2}.$$

This function has the maximum value as $2\theta-\alpha=\frac{\pi}{2}$. And hence this value

$f_m=\frac{1-\sin\alpha}{2}$. Hence, from (3) the maximum range $R_m=\frac{2V^2}{g\cos^2\alpha}\cdot\frac{1-\sin\alpha}{2} \Rightarrow$

$$R_m=\frac{V^2}{g}\cdot\frac{1-\sin\alpha}{1-\sin^2\alpha} \Rightarrow R_m=\frac{V^2}{g(1+\sin\alpha)}.$$

10 Solution



Initial conditions when $t=0$

$$x=0,\ y=h;$$

$$\dot{x}=V\cos\alpha;\quad \dot{y}=V\sin\alpha.$$

Point O is the foot of the cliff, α is the angle of elevation.

After t seconds the particle is at position

$$x=V\cos\alpha\cdot t, \tag{1}$$

$$y = h + V\sin\alpha \cdot t - \frac{g t^2}{2}. \tag{2}$$

When the particle hits the ground, $y = 0$. Hence from (2) for the time, when $y = 0$,

$t^2 - \frac{2V}{g}\sin\alpha \cdot t - \frac{2h}{g} = 0$, $t = \frac{V\sin\alpha}{g} + \sqrt{\left(\frac{V\sin\alpha}{g}\right)^2 + \frac{h}{g}}$,

$t = \frac{1}{g}\left(V\sin\alpha + \sqrt{V^2\sin^2\alpha + gh}\right)$.

Substituting this value of t into (1), we obtain the horizontal distance covered before landing in the sea: $x = \frac{V\cos\alpha}{g}\left(V\sin\alpha + \sqrt{(V\sin\alpha)^2 + gh}\right)$.

Let the function $f(\alpha) = \cos\alpha\left(\sin\alpha + \sqrt{\sin^2\alpha + a}\right)$, (3)

Where $a = \frac{gh}{V^2}$. Then $x = \frac{V^2}{g}f(\alpha)$.

It is easy to see that $f(\alpha) = \frac{\sin 2\alpha}{2} + \sqrt{\left(\frac{\sin 2\alpha}{2}\right)^2 + a\cos^2\alpha}$.

The derivative of this function is $f'(\alpha) = \cos 2\alpha + \dfrac{\sin 2\alpha\cos 2\alpha - a\sin 2\alpha}{2\sqrt{\left(\frac{\sin 2\alpha}{2}\right)^2 + a\cos^2\alpha}}$.

If we want to find the maximum value of $f(\alpha)$ and hence the greatest horizontal distance required, we must solve the equation $f'(\alpha) = 0$:

$2\cos 2\alpha \cdot \sqrt{\left(\frac{\sin 2\alpha}{2}\right)^2 + a\cos^2\alpha} = \sin 2\alpha\cos 2\alpha - a\sin 2\alpha$.

Squaring, $4\cos^2 2\alpha\left\{\left(\frac{\sin 2\alpha}{2}\right)^2 + a\cos^2\alpha\right\} = \sin^2 2\alpha(\cos 2\alpha - a)^2$;

$\sin^2 2\alpha\cos^2 2\alpha + 4a\cos^2 2\alpha\cos^2\alpha = \sin^2 2\alpha\cos^2 2\alpha - 2a\sin^2 2\alpha\cos 2\alpha + a^2\sin^2 2\alpha$;

$4\cos^2 2\alpha\cos^2\alpha = a\sin^2 2\alpha - 2\sin^2 2\alpha\cos 2\alpha$. But $\cos 2\alpha = 2\cos^2\alpha - 1$, hence

$4\cos^2 2\alpha\cos^2\alpha = a\sin^2 2\alpha - 2\sin^2 2\alpha(2\cos^2\alpha - 1)$,

$4\cos^2\alpha\left(\cos^2 2\alpha + \sin^2 2\alpha\right) = (a+2)\sin^2 2\alpha$, $\frac{4}{a+2}\cos^2\alpha = (2\sin\alpha\cos\alpha)^2$,

$\sin^2\alpha = \frac{1}{a+2} \Rightarrow \cos^2\alpha = \frac{a+1}{a+2}$.

Hence from (3) the maximum value of $f(\alpha)$ is $f_m = \sqrt{\dfrac{a+1}{a+2}}\left(\dfrac{1}{\sqrt{a+2}} + \sqrt{\dfrac{1}{a+2} + a}\right)$,

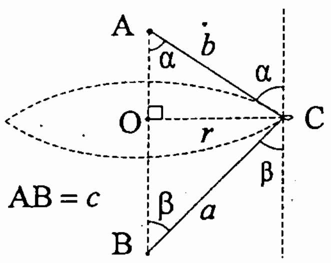
$f_m = \sqrt{\dfrac{a+1}{a+2}}\left(\dfrac{1}{\sqrt{a+2}} + \dfrac{a+1}{\sqrt{a+2}}\right)$, $f_m = \sqrt{\dfrac{a+1}{a+2}} \cdot \sqrt{a+2} = \sqrt{a+1}$; $a = \dfrac{gh}{V^2} \Rightarrow$

$f_m = \dfrac{1}{V}\sqrt{V^2 + gh}$. But $x = \dfrac{V^2}{g} f(\alpha)$. Hence the greatest horizontal distance

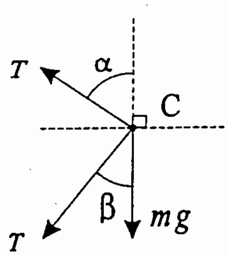
$x_m = \dfrac{V}{g}\sqrt{V^2 + gh}$.

11 Solution

Dimension diagram



Forces on C



The resultant force is $m\omega^2 r$ horizontally to the left. The vertical component is zero, hence

$T\cos\alpha - T\cos\beta = mg$ (1)

The horizontal component is $m\omega^2 r$,

hence $T\sin\alpha + T\sin\beta = m\omega^2 r$ (2)

From (1) $T = \dfrac{mg}{\cos\alpha - \cos\beta}$, analogously from (2) $T = \dfrac{m\omega^2 r}{\sin\alpha + \sin\beta}$. Equating these expressions for T, $g(\sin\alpha + \sin\beta) = \omega^2 r(\cos\alpha - \cos\beta)$. But $r = a\sin\beta$, and dividing the last equation by ab, we obtain $g\left(\dfrac{\sin\alpha}{a}\cdot\dfrac{1}{b} + \dfrac{\sin\beta}{b}\cdot\dfrac{1}{a}\right) = \omega^2\dfrac{\sin\beta}{b}(\cos\alpha - \cos\beta)$.

For only triangle $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$, hence $g\left(\dfrac{1}{b} + \dfrac{1}{a}\right) = \omega^2(\cos\alpha - \cos\beta)$. (3)

In the triangle ABC we have $a^2 = b^2 + c^2 - 2bc\cdot\cos\alpha$, $b^2 = a^2 + c^2 - 2ac\cdot\cos\beta \Rightarrow$

$\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$, $\cos\beta = \dfrac{a^2 + c^2 - b^2}{2ac}$. Substituting $\cos\alpha$ and $\cos\beta$ into (3),

$$g\frac{(a+b)}{ab} = \omega^2\left\{\frac{b^2 + c^2 - a^2}{2bc} - \frac{a^2 + c^2 - b^2}{2ac}\right\},$$

$$2gc(a+b) = \omega^2\left\{a\left(b^2 + c^2 - a^2\right) - b\left(a^2 + c^2 - b^2\right)\right\},$$

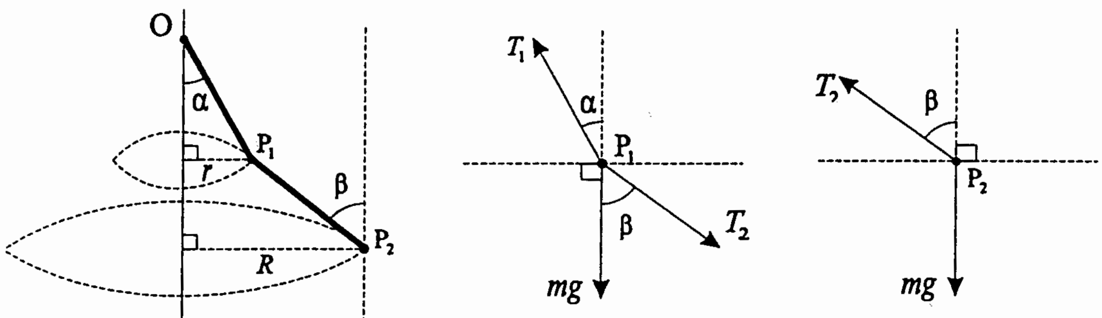
$$2gc(a+b)=\omega^2\left\{c^2(a-b)+\left(b^3-a^3\right)+\left(ab^2-ba^2\right)\right\},$$

$$2gc(a+b)=\omega^2\left\{c^2(a-b)-(a-b)\left(a^2+ab+b^2\right)-(a-b)ab\right\},$$

$$2gc(a+b)=\omega^2(a-b)\left\{c^2-(a+b)^2\right\}.$$

12 Solution

Dimension diagram | Forces on P_1 | Forces on P_2



$OP_1 = P_1P_2 = l$

The resultant force on P_1 is $m\omega^2 r$ horizontally to the left, where $r = l\sin\alpha$. The vertical component is zero and the horizontal one is $m\omega^2 r$. Hence we have

$$T_1\cos\alpha - T_2\cos\beta = mg, \tag{1}$$

$$T_1\sin\alpha - T_2\sin\beta = m\omega^2 l\sin\alpha. \tag{2}$$

The resultant force on P_2 is $m\omega^2 R$ horizontally to the left, where $R = l(\sin\alpha+\sin\beta)$.

The vertical component is zero and the horizontal one is $m\omega^2 R$. Hence we have

$$T_2\cos\beta = mg, \tag{3}$$

$$T_2\sin\beta = m\omega^2 l(\sin\alpha+\sin\beta). \tag{4}$$

(a) Substituting (3) into (1), $\quad T_1\cos\alpha = 2mg,$ (5)

and substituting (4) into (2), $\quad T_1\sin\alpha = m\omega^2 l(2\sin\alpha+\sin\beta).$ (6)

Dividing (6) by (5), $\tan\alpha = \dfrac{l\omega^2}{g}\left(\sin\alpha+\dfrac{1}{2}\sin\beta\right).$

(b) Dividing (4) by (3), $\tan\beta = \dfrac{l\omega^2}{g}(\sin\alpha+\sin\beta).$